NOTE ON DEDUCED PROBABILITY DISTRIBUTIONS

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In this Bulletin, December, 1936, A. H. Copeland* resumed the study of the problem first suggested by H. Poincaré: How can the fact of uniform probability distribution, which we meet so frequently in different games of chance, be explained? Recently E. Hopf devoted a profound essay† to this question and he has just published a short note‡ dealing with his principal results. I want to contribute a quite simple remark which seems to show how far the results are independent of the particular form of dynamical equations.

We assume that there exists a density function $f(x)$ for the one-dimensional variable x, such that $\int_a^b f(x)dx$ denotes the probability that the value of x falls in the interval (a, b) and $\int_{-\infty}^{\infty} f(x)dx = 1$. If between x and y there is established a one-to-one correspondence

$$(1) \qquad y = y(x), \qquad x = x(y),$$

the given density function $f(x)$ leads to a new density function $g(y)$ defined by

$$(2) \qquad g(y) = f(x)\frac{dx}{dy}.$$

The integral $\int_a^b g(y)dy$ gives, of course, the probability that y belongs to the interval (a, b) and $\int_{-\infty}^{\infty} g(y)dy = 1$.

Now we suppose y to be an "angular" variable, that is, instead of y we consider the new variable:

$$(3) \qquad \eta = y - [y], \qquad 0 \leqq \eta < 1,$$

and try to determine the probability distribution $\phi(\eta)$ of η. Evidently, if ν is a positive or negative integer, the probability density of η is given by

$$(4) \qquad \phi(\eta) = \sum_\nu g(\eta + \nu) = \sum_\nu f(x_\nu)\left(\frac{dx}{dy}\right)_{x=x_\nu}; \qquad x_\nu = x(\eta + \nu).$$

* Vol. 42, p. 895.

† Journal of Mathematics and Physics, Massachusetts Institute of Technology, vol. 13 (1934).

‡ Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 46 (1936), p. 179.

We may suppose the transformation ratio $dx/dy>0$. In this case the consecutive values $\cdots x(-2), x(-1), x(0), x(1), x(2), \cdots$ define an infinite set of intervals corresponding to intervals of length 1 on the y-axis. Let η' and η'' be two values of η; then the corresponding values $x_\nu'=x(\eta'+\nu)$ and $x_\nu''=x(\eta''+\nu)$ fall in the same interval $(x(\nu), x(\nu+1))$. Therefore, the difference between the values of the products

$$f(x_\nu')\left(\frac{dx}{dy}\right)_{x=x_\nu'} \quad \text{and} \quad f(x_\nu'')\left(\frac{dx}{dy}\right)_{x=x_\nu''},$$

is less than or equal to the variation of the product $f\cdot dx/dy$ through the interval $(x(\nu), x(\nu+1))$, and the difference between the two values of the sum (4) for η' and η'' does not exceed the value of the total variation of the same product. Hence, our theorem follows:

The maximum difference between two values of the deduced probability density $\phi(\eta)$ is less than or equal to the total variation of the product of initial density $f(x)$ by the transformation ratio dx/dy.

If we consider an infinite set of similar problems where the initial distribution $f(x)$ remains unchanged and the transformation ratio is multiplied by a parameter λ, then the deduced distribution $\phi(\eta)$ approaches uniformity as the parameter λ approaches 0 and the functions $f(x)$ and dx/dy are of finite variation.

The mechanical example mentioned by Copeland and by Hopf consists in a system rotating about a vertical axis and subjected to friction forces which depend on the instantaneous angular velocity ω. The dynamical equation is given by

$$\frac{d\omega}{dt} = -r(\omega). \tag{5}$$

Let x be the initial value ω_0 of ω. Until the system comes to rest, a point at the distance 1 from the axis will travel a distance which may be designed by $2\pi y$. Then it follows from (5) that

$$2\pi y = \int \omega dt = \int_0^x \frac{\omega d\omega}{r(\omega)}, \qquad \frac{dx}{dy} = 2\pi\frac{r(x)}{x}. \tag{6}$$

Copeland supposes the friction $r(x)$ to be proportional to a parameter λ, but the distribution $f(x)$ to be independent of λ. In this case it is clear that, in consequence of (6), $f(x)\cdot dx/dy$ approaches zero with $\lambda\rightarrow 0$, and our theorem shows that the asymptotic value of $\phi(\eta)$ is a constant.

On the other hand, Hopf considers the initial distribution to be given in the form $f(x)=f_1(x-\lambda)$, where f_1 is a function of one variable and λ a parameter. Moreover, he supposes that

$$\lim_{x\to\infty}\frac{r(x)}{x}=0. \tag{7}$$

We find $f(x)\cdot dx/dy=2\pi f_1(x-\lambda)\cdot r(x)/x=2\pi f_1(z)r(\lambda+z)/(\lambda+z)$, which approaches zero, according to (7), as λ increases. If the functions f_1 and r/x are of finite variation, it follows from our theorem that $\phi(\eta)$ approaches uniformity.

It is a quite different question to decide whether the foregoing investigation is or is not sufficient to explain the fact of the nearly perfect uniformity of distribution in a particular case of a real game. Let us consider a sort of roulette consisting of a billiard ball which runs in a smooth circular channel subjected to a constant resistance $r(\omega)=c$; the number of revolutions is found to vary from about 8.1 to 12.1. Our equation (6) gives

$$2\pi y=\frac{x^2}{2c},\qquad \frac{dx}{dy}=\frac{2\pi c}{x}=\left(\frac{\pi c}{y}\right)^{1/2},\qquad x=2(\pi c y)^{1/2}.$$

Therefore x varies from $9(0.4\ \pi c)^{1/2}$ to $11(0.4\ \pi c)^{1/2}$, and if we assume $f(x)$ to be constant in this interval of length $2(0.4\ \pi c)^{1/2}$, we find

$$g(y)=\frac{1}{2(0.4\ \pi c)^{1/2}}\left(\frac{\pi c}{y}\right)^{1/2}=\frac{1}{2(0.4\ y)^{1/2}},\qquad 8.1\leqq y\leqq 12.1.$$

The resulting density function $\phi(\eta)$ is a monotonic decreasing function in the interval from $\eta=0.1$ to $\eta=1.1$. If we divide the whole circle in two parts from $\eta=0.1$ to $\eta=0.6$ and from 0.6 to 1.1, it follows that the probability of a rest position in the first of these semicircles is

$$\int_{8.1}^{8.6}g(y)dy+\int_{9.1}^{9.6}\cdots+\int_{10.1}^{10.6}\cdots+\int_{11.1}^{11.6}\cdots=0.506.$$

The excess of 1.2% is doubtless too large for a fair game of chance. It seems that in such cases other circumstances increase the tendency towards uniformity.

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