Les Fonctions Polyharmoniques, by Miron Nicolesco. (Actualités Scientifiques et Industrielles, no. 331; exposés sur la théorie des fonctions, IV.) Paris, Hermann, 1936.

This monograph is the fourth in a series devoted to recent developments in the theory of functions, which is being published under the editorship of Professor Montel of Paris. These monographs have as an objective the presentation of synthetic summaries of some of the advances made during the last few years; no attempt is made at giving a detailed exposition, but a clear-cut sketch of the developments, supplemented by suitable bibliographies, is made available to the reader who wishes to orient himself in the fields treated.

The polyharmonic functions of order p (or p-harmonic), in an n-dimensional space, are those functions $U(x_1, x_2, \cdots, x_n)$ of n independent variables which satisfy the partial differential equation $\Delta^p U=0$, where Δ^p is the pth iteration of the Laplacian operator Δ. Thus,

$$\Delta \equiv \sum_{s=1}^{n} \frac{\partial^2}{\partial x_s^2}, \ \Delta^k = \Delta(\Delta^{k-1}), \ (k = 1, 2, 3, \cdots), \ \Delta^0 \equiv 1.$$

For $p=1$, we have the case of the ordinary harmonic functions. The biharmonic case ($p=2$), being of importance in the theory of elasticity, was the object of research rather early. Thus, Airy introduced the stress function from which the components of the stress tensor may easily be calculated, and showed that it had to satisfy the equation $\Delta^2\phi=0$, with appropriate boundary conditions.

The earlier researches on polyharmonic functions centered, quite naturally, about the study of boundary value problems analogous to the classical ones in the theory of harmonic functions. Most of the results obtained in this direction are due to the Italian school of mathematicians led by Almansi, Volterra, Marcolongo, Lauricella, Boggio, and others. In these researches, the investigation of the intrinsic, structural properties of the polyharmonic functions was, for the most part, subsidiary to the solution of the particular boundary value problems being studied. Even so important a result as Almansi's expansion theorem was treated primarily from the point of view of its bearing on this type of problem.

Since 1930, Professor Nicolesco and others have undertaken a systematic study of the intrinsic properties of the polyharmonic functions for their own sake and have succeeded in filling important gaps in the theory of these interesting functions. The main purpose of his monograph is to give a synthesis of the results so far obtained. The work also contains a brief summary of some of the work done in connection with these functions and their associated boundary value problems, placing some emphasis on the biharmonic case of elasticity theory.

An extensive memoir by Professor Nicolesco, entitled *Recherches sur les fonctions polyharmoniques*, to appear shortly in the Annales de l'École Normale Supérieure, as a supplement to the present work, should be of the greatest interest.

M. A. BASOCO

Les Conditions de Monogénéité. By D. Menchoff. (Actualités Scientifiques et Industrielles, no. 329.) Paris, Hermann, 1936. 53 pp.

This monograph, as its title implies, is a report on some of the recent developments in the direction of obtaining sufficient conditions that a function of a complex variable be monogenic at a point or holomorphic in a region. Monogeneity at a point being equivalent to the existence of a finite-valued derivative at the point, the initial theorem quoted gives also the simplest condition, which is that the functions u and v, where $f=u+iv$, possess a Stolz differential and satisfy the Cauchy-Riemann differential equations. If the Stolz differential is assumed, then it is also sufficient either (a) that the difference quotient $\Delta f/\Delta z$ have the same limit for any two distinct directions, or (b) that arg $\Delta f/\Delta z$ have the same limit for three distinct directions, or (c) that $|\Delta f/\Delta z|$ have the same limit in three directions, but in the latter case $\bar{f}$ may be monogenic instead of f. The theorems giving sufficient conditions for the holomorphism of a function in a region depend upon sufficient conditions for the expression of the integral $\int f(z)dz$ around a rectangle with sides parallel to the axes in the form of the double integral

$$-\iint\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)dx\,dy+i\iint\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)dx\,dy$$

and the fact that if $\overline{\lim}\ |\Delta f/\Delta z|$ is finite on a measurable set, then the Stolz differential of f exists except on a set of measure zero, various theorems being obtained by giving sufficient conditions for the application of these results and the use of Morera's theorem.

In the main the monograph is a brief presentation of the author's investigations in these questions, which may be justifiable, but involves the possibility of overlooking simplifications in presentation. For instance, the three supplementary sufficient conditions for the monogeneity of a function at a point are geometrically intuitive if use is made of the Kasner circle.* However, the monograph is informative and suggestive; especially might one call attention to the remark in the introduction that, while many theorems have a form which involves only the complex variable situation, it has so far been necessary to use in their proof deep-seated methods of the modern theory of real functions, and that it would be interesting and desirable to derive these same theorems without departing from the setting in which they are stated.

T. H. HILDEBRANDT

Théorie de l'Addition des Variables Aléatoires. By P. Lévy. (Monographies des Probabilités, publiés sous la direction de E. Borel, no. 1.) Paris, Gauthier-Villars, 1937. 17+328 pp.

Professor Lévy's book is the first of a series of monographs on probability and its applications. This first monograph covers the field of Lévy's special interest, to which he has made valuable and often definitive contributions: certain questions connected with asymptotic problems in probability theory. The first four chapters include a general introduction to probability, designed to make the book complete in itself. Chapter V contains theorems related to the Gaussian law—Cramer's recently proved theorem and various extensions of the Liapounoff theorem. In Chapter VI, Lévy discusses series whose terms are mutually independent chance variables. He includes an important theorem on dispersion, recently derived by himself and Doeblin. Chapter VII considers a chance variable x_t (depending on the parameter t) whose increments in non-overlapping t-intervals are independent. The treatment is somewhat confusing in that measurability considerations are omitted and it is not clear whether x_t considered as a function of t has been proved to have (almost certainly), at worst, jumps at its points of discontinuity. Lévy gives a considerably simplified version of

* This Bulletin, vol. 34 (1928), p. 561.