

ON REFLECTION OF SINGULARITIES OF HARMONIC
FUNCTIONS CORRESPONDING TO THE
BOUNDARY CONDITION $\partial u/\partial n + au = 0$

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1. *Introduction.* Familiar "reflection" principles across a plane at which a harmonic function u satisfies either of the two boundary conditions

$$(1) \quad u = 0,$$

$$(2) \quad \partial u/\partial n = 0,$$

where $\partial/\partial n$ denotes the normal derivative, extend the function u from one side of the plane to the other one by means of its negative or positive image respectively. In particular, the singularities of u to one side of the plane are also reflected into their negative or positive images.

In the following we consider the nature of the "reflection" or analytic continuation of a harmonic function u across a plane boundary corresponding to what is perhaps the next simplest boundary condition, namely:

$$(3) \quad \frac{\partial u}{\partial n} + au = 0,$$

where a is a constant. It is shown that the image of each singularity S_0 of u is relatively complex and consists of

- (a) a positive image S_1 of S_0 in the boundary plane;
- (b) an exponential trail of negative images along the line through S_0 and S_1 , beyond S_1 , and totalling in amount double the negative of S_0 .

Results similar to the above are established for other differential equations; for instance, for the equation of heat conduction. Conditions with higher order derivatives are also considered.

Aside from the interest of the subject matter in connection with analytic continuation, as well as from the point of view of general curiosity that makes one "peep behind the looking glass," the subject is also of interest in view of several pos-

sible applications. One application is to the problem of heat flow from an underground source; another one is to the problem of eddy currents induced in a semi-infinite solid with a plane boundary by nearby alternating currents. These applications are discussed.

2. *Analytic Continuation of Harmonic Functions Corresponding to (3).* Consider a function u , harmonic for $x \geq 0$ and satisfying the boundary condition:

$$(4) \quad \frac{\partial u}{\partial x} - au = 0 \quad \text{along } x = 0.$$

Let

$$(5) \quad w = \frac{\partial u}{\partial x} - au.$$

The function w is also harmonic, while along $x=0$ it vanishes. Hence w may be continued analytically to $x < 0$ by means of negative reflection:

$$(6) \quad w(-x, y, z) = -w(x, y, z).$$

Substitution from (5) converts (6) into a differential equation whose solution yields

$$(7) \quad u(-x) = -u(x) + 2a \int_0^x e^{a(x'-x)} u(x') dx'.$$

This is what the simple negative and positive reflection corresponding to (1) and (2) respectively is to be replaced by in case of (4).

3. *Green's Function Corresponding to (4).* To examine the nature of reflection of the singularities of harmonic functions satisfying (4), consider for $x > 0$ a function u which satisfies the following requirements:

$$(8) \quad \left\{ \begin{array}{l} \text{(a) } u \text{ is harmonic except near } P_0 \equiv (h, 0, 0). \\ \text{(b) near } P_0, u \text{ is of the form } u = 1/r_0 + u', \text{ where } r_0 \text{ is the} \\ \quad \text{distance from } P_0, \text{ and } u' \text{ is harmonic.} \\ \text{(c) } u \text{ and its first derivatives vanish at infinity.} \\ \text{(d) along } x = 0, u \text{ satisfies the boundary condition (4).} \end{array} \right.$$

This function u may be designated briefly as the "Green's function" for $x > 0$ with pole at P_0 , corresponding to the boundary condition (4).

If the real part of a is non-negative,

$$(9) \quad R(a) \geq 0,$$

a solution of (8) is given by

$$(10) \quad v = 1/r_0 + 1/r_1 - 2aI,$$

where

$$(11) \quad I = \int_{-\infty}^{-h} (e^{a(h+x')}/r') dx';$$

here r_1 is the distance to $P_1 \equiv (-h, 0, 0)$ and r' the distance to the point $(x', 0, 0)$ on the x -axis. If we use an electrostatic terminology which designates $1/r_0$ as the potential of a unit charge at P_0 , the solution v may be described as the potential of unit charges at P_0 and P_1 and of a line distribution of charge of density $-2a e^{a(x+h)}$ per unit length of the x -axis, extending from $x = -\infty$ to P_1 . The total charge of the line distribution is $-2a \int_{-\infty}^{-h} e^{a(x+h)} dx = -2$. Added to the unit charges at P_0 and P_1 this results in a net charge zero. Thus v vanishes at infinity and to a higher order than $1/r$.

To prove that v satisfies the boundary condition (4), one proves

$$\frac{\partial I}{\partial x} = -1/r_1 + aI$$

by differentiating under the integral sign, replacing $\partial(r^{-1})/\partial x$ by $-\partial(r^{-1})/\partial x'$, and integrating by parts. It follows that

$$(12) \quad \frac{\partial v}{\partial x} - av = a \left(\frac{1}{r_1} - \frac{1}{r_0} \right) + \frac{\partial}{\partial x} \left(\frac{1}{r_0} + \frac{1}{r_1} \right).$$

Since the latter vanishes at $x=0$ the proof is complete.

4. *Ways of Arriving at the Preceding Green's Function. Its Uniqueness.* Introduce the function w given by (5). The latter is harmonic for $x > 0$ except at P_0 where it becomes infinite like $\partial(1/r_0)/\partial x - a/r_0$; it vanishes at infinity and also vanishes at $x=0$. It follows then that u satisfies (12):

$$(13) \quad w = \frac{\partial u}{\partial x} - au = \left(\frac{\partial}{\partial x} - a \right) \left(\frac{1}{r_0} \right) + \left(\frac{\partial}{\partial x} + a \right) \frac{1}{r_1}.$$

Integration of (13) in the form

$$(14) \quad u = - \int_x^\infty e^{a(x-x')} w(x', y, z) dx' = \int_0^\infty e^{-as} w(x+s, y, z) ds,$$

and consequent integration by parts and simplification, lead to (10), (11).

Another way of obtaining u —and this is the way in which it was first obtained—is by means of the Fourier or Fourier-Bessel integral. This will be found useful for the case in which a does not satisfy (9). In the latter case it will be seen that while (13) still persists, the integrals in (11), (14) are divergent.

Write u in the form

$$(15) \quad u = \frac{1}{r_0} + \int_0^\infty f(\lambda) e^{-\lambda x} J_0(\lambda \rho) d\lambda,$$

where $\rho^2 = y^2 + z^2$; utilizing the familiar integral expansion

$$(16) \quad 1/r = (\rho^2 + x^2)^{-1/2} = \int_0^\infty J_0(\lambda \rho) e^{-\lambda |x|} d\lambda,$$

and letting the resultant integrand of u satisfy (4), one obtains

$$(17) \quad f(\lambda) = e^{-\lambda h} - e^{-\lambda h} \frac{2a}{\lambda + a}.$$

The first term in (17) leads to the positive image at P_1 . The second term yields

$$(18) \quad -2a \int_0^\infty \frac{e^{-\lambda(x+h)} J_0(\lambda \rho) d\lambda}{\lambda + a}.$$

By deforming the path of integration in the complex λ -plane the analytic continuation of u to $x < 0$ may be obtained and a charge density $-2a e^{a(x+h)}$ deduced along the negative x -axis for $x < -h$. For negative a the path of integration in (18) must avoid the pole $\lambda = -a$ and hence cannot be confined to the real axis. We shall not stop over the particulars. The analytic continuation of harmonic functions by deforming the path of integration of proper integrals will be considered at a future date.

It is worth noting that for negative a the solution of (8) is no longer unique, since many harmonic solutions of the homogeneous equation $\partial u/\partial x - au = 0$ (see (13)) exist which are harmonic and vanish at infinity for $x > 0$, namely,

$$(19) \quad C(y, z)e^{ax},$$

where C is a solution of

$$(20) \quad \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2} + a^2 C = 0.$$

Thus, when we confine ourselves to axial symmetry, u is arbitrary to within a multiple of

$$(21) \quad J_0(a\rho)e^{ax},$$

which will be recognized as proportional to the residue of the integrand (18) at the pole $\lambda = -a$.

For $R(a) \geq 0$ the Green's function vanishes at infinity for $x \geq 0$; for $R(a) < 0$, if the vanishing at infinity (condition (8d)) is to be interpreted as applying for all x , then *no* solution of (8) exists.

5. *The Reflection of Singularities of Other Harmonic Functions.*

The singularities of other single-valued harmonic functions can be obtained by reflections similar to that of the Green's function. For instance, the harmonic function which becomes singular at P_0 like $\partial(1/r_0)/\partial y$ (while satisfying (8a), (8b), (8d)) is equal to $\partial v/\partial y$, where v is given by (10). Describing the singularity at P_0 as a dipole of doublet of unit moment whose axis is in the direction of the y -axis, we see that the analytic continuation to $x < 0$ has a similar dipole at P_1 and an exponential trail of dipoles along the x -axis beyond P_1 . This follows from the fact that if u satisfies (4), so does also $\partial u/\partial y$, or it may be proved by replacing the derivative by a limit of a difference quotient, interpreting each of the terms of the difference as a potential of a point charge, applying the now familiar reflection to it, and passing to the limit.

The harmonic function u which becomes singular like $\partial(1/r_0)/\partial x$ at P_0 is *not* given by $\partial v/\partial x$ where v is given by (10) since, if u satisfies (4), $\partial u/\partial x$ will not in general satisfy it; this harmonic function may, however, be obtained by the limit-

ing process which analyzes the singularity at P_0 into singularities of point charges. There results the formula

$$(22) \quad u = \frac{\partial\left(\frac{1}{r_0}\right)}{\partial x} - \frac{\partial\left(\frac{1}{r_1}\right)}{\partial x} - \frac{2a}{r_1} + 2a^2 \int_{-\infty}^{-h} \frac{e^{a(x+h)}}{r'} dx',$$

exhibiting for u a dipole at P_1 but of opposite moment to that at P_0 , a point charge of amount $-2a$, and an exponential trail of positive charges.

Harmonic functions which satisfy (4) but fail to satisfy Laplace's equation over curves, surfaces, or regions by behaving like the potentials of distributions of charges over these respective loci, are similarly treated for reflection across $x=0$ by replacing the integrals by finite sums, reflecting the point charge singularity of each term, and passing to a limit.

6. *The Two-Dimensional Green's Function Corresponding to (4).* The two-dimensional case can be treated similarly to the three-dimensional one. The condition (8b) is replaced by the requirement that, near $P_0 \equiv (h, 0)$, u is to become infinite like $-\ln r_0 = -\ln [(x-h)^2 + y^2]^{1/2}$; the requirement (8c) at infinity might be kept, though the condition $u = O(\ln r)$ in its place would be suggested by the theory of the logarithmic potential. One proves similarly that

$$(23) \quad u = -\ln r_0 - \ln r_1 + 2a \int_{-\infty}^{-h} \ln r' e^{a(h+x')} dx'.$$

The interpretation in terms of a positive image and an exponential trail of negative images is obvious.

For the present (two-dimensional case) the representation of u as the potential of a charge distribution is not at all unique. We shall prove this for real positive a .

Indeed, by integration by parts and introducing imaginaries, (23) may be put in the form

$$u = R \left[\ln \frac{z+h}{z-h} + 2 \int_{-\infty}^{-h} \frac{[e^{a(h+x')}/(z-x')] dx'}{z-x'} \right],$$

where $z = x + iy^*$ and " R " denotes "the real part of." Replacing

* No confusion is likely to arise from the use of z in a sense different from its previous one.

the (dummy) variable of integration x' by z' , one may deform the path of integration from the real axis into an arbitrary curve C between the same limits. Thus

$$(24) \quad u = R \left[\ln \frac{z+h}{z-h} + 2 \int_C \frac{e^{a(h+z')}}{z-z'} dz' \right],$$

$$(24') \quad u = R \ln \frac{z+h}{z-h} + 2 \int_C R[e^{a(h+z')}] R \left(\frac{dz'}{z-z'} \right) - 2 \int_C I[e^{a(h+z')}] I \left(\frac{dz'}{z-z'} \right),$$

where $I(z)$ refers to the "imaginary part of z ." Now the real as well as the imaginary part of $1/(z-z')$ represents a potential of a doublet or dipole placed at $z=z'$. Hence the curve integrals in (24') can be interpreted as the potential of a proper distribution of (logarithmic) dipoles over C , with axes respectively normal to and along C . The tangential dipoles can by integration by parts be reduced to a distribution of poles.

Of the various charge distributions the one derived from (23) is, no doubt, the simplest one.

By integrating (24) one obtains

$$(25) \quad u = R \left\{ \ln \frac{z+h}{z-h} - 2e^{a(h+z)} Ei[-a(h+z)] \right\},$$

where Ei denotes the "integral exponential." The latter is multiple-valued. Another way of proving that u may be considered to be the potential of a variety of distributions is by rendering Ei single valued by using a branch corresponding to a cut along the curve C . The resulting discontinuities in C and its normal derivative, when divided by 2π , yield the densities of normal doublets and of poles respectively.

The Fourier integral (in y) of u is also of some interest. Starting with the form (24) and utilizing

$$\frac{1}{z} = \int_0^\infty e^{-z\lambda} d\lambda \quad \text{for } R(z) > 0,$$

one obtains

$$\begin{aligned}
 (26) \quad u &= R \left[\ln \frac{z+h}{z-h} - 2 \int_0^\infty \frac{e^{-(h+z)\lambda}}{a+\lambda} d\lambda \right] \\
 &= \ln \frac{r_2}{r_1} - 2 \int_0^\infty \frac{e^{-\lambda(x+h)} \cos \lambda y}{a+\lambda} d\lambda.
 \end{aligned}$$

The Fourier integral as well as (25), (26) apply for $R(a) \leq 0$, as well.

7. *Other Differential Equations. Applications.* As stated in the introduction, the reflection of singularities across a plane boundary corresponding to the boundary condition (3), in terms of a positive image and an exponential trail of negative images, can be carried over to certain other differential equations. Among these are

$$(27) \quad \nabla^2 u - \lambda u = 0, \quad \lambda = \text{constant},$$

$$(28) \quad \partial u / \partial t = k \nabla^2 u.$$

The proof is again carried out by forming the function w given by (5), reflecting it, then solving for u .

For (28), the heat conduction equation, the boundary condition (4) represents a loss of heat (for real positive a) at the plane boundary $x=0$ at a rate proportional to the boundary temperature. If we consider singularities due to instantaneous or permanent "heat sources" (while (4) applies), the temperature can in every case be determined by using a similar image source as well as an exponential trail of "sinks."

For steady point or line heat sources, (10) and (25) essentially still apply since (28) reduces to Laplace's equation. As an example of an instantaneous source recall that

$$(29) \quad v = (4\pi kt)^{-1} e^{-(x^2+y^2)/4kt} \text{ for } t > 0, \quad v = 0 \text{ for } t < 0,$$

represents the temperature in an infinite medium due to an instantaneous line source of heat at the time $t=0$ along the z -axis, the amount of heat discharged being so chosen that $\iint v dx dy = 1$ for $t > 0$.* The temperature due to a similar line source at $(h, 0)$ when (4) holds is given by

* See Carslaw, *The Mathematical Theory of Conduction of Heat in Solids*, Chap. IX.

$$(30) \quad 4\pi k t u = e^{-[(x-h)^2+v^2]/4kt} + e^{-[(x+h)^2+v^2]/4kt} \\ - 2a \int_{-\infty}^{-h} e^{a(x'+h)} e^{-[(x+x')^2+h^2]/4kt} dx'.$$

The last integral can be expressed in terms of the "complementary error" function defined by: $\operatorname{erfc}(x) = (2/\pi^{1/2}) \int_x^{\infty} e^{-u^2} du$.

Another application of the boundary condition (3) that will now be considered is in connection with eddy currents excited in a conducting semi-infinite solid $x < 0$ by alternating currents flowing in free space in the region $x > 0$. Time is assumed to enter as a factor $e^{i\omega t}$. If angular frequency ω is not too high, displacement currents may be neglected; then electromagnetic field theory, as expressed, say, by Maxwell's equations, shows that the components of both the magnetic and electric field in the solid conductor satisfy the differential equation

$$(31) \quad \nabla^2 u = \alpha^2 u,$$

where

$$(32) \quad \alpha = (1+i)\beta, \quad \beta = (2\pi\omega\lambda\mu)^{1/2}, \quad \lambda = \text{conductivity}, \quad \mu = \text{permeability}.$$

In free space the field components are harmonic in regions free from currents; near each current element $d\mathbf{i} = (l, m, n) |i| ds$, the electric and magnetic fields become infinite respectively like

$$(33) \quad \mathbf{E} = 4\pi i \omega d\mathbf{i}/r, \\ \mathbf{H} = 4\pi \nabla \times (d\mathbf{i}/r) \\ (34) \quad = 4\pi |i| \left[\left(m \frac{\partial}{\partial y} - n \frac{\partial}{\partial z} \right), (\dots), (\dots) \right] (1/r),$$

where r is the distance to the current element, while l, m, n are its direction cosines. At the boundary $x=0$ the tangential field components $H_y, H_z; E_y, E_z$ are continuous.

We shall be concerned with the asymptotic behavior of the solution for large β . It will be shown that for β large the differential equation (31) can be replaced by proper boundary conditions on the components of the harmonic functions \mathbf{E}, \mathbf{H} .

By examining cases where a complete solution is available it is found that for large β or large $|\alpha|$, the terms $\partial^2 u / \partial y^2, \partial^2 u / \partial z^2$ in (31) are negligible in comparison with the remaining terms, so that (31) approximately reduces to

$$(35) \quad \frac{\partial^2 u}{\partial x^2} = \alpha^2 u,$$

whose pertinent solutions are

$$(36) \quad u = u(y, z)e^{\alpha x}.$$

Physically, the constant $1/\beta$ has the dimensions of a length, and when this length is small (compared to the dimensions of the exciting circuit and its distance from the plane $x=0$), the field developed in the conducting solid decays rapidly with penetration into the solid and the resulting currents are confined to a thin layer near the boundary thus exhibiting the phenomena of "skin effect." Using the above approximation we see then that within $x < 0$ all the field components are of the form (36), where the coefficient of $e^{\alpha x}$ refers to their values at the boundary of the solid. Applying now the boundary conditions at $x=0$, one proves readily that the normal magnetic field H_x satisfies the boundary condition

$$(37) \quad \left(\frac{\partial}{\partial x} - \frac{\alpha}{\mu} \right) H_x = 0 \quad \text{at} \quad x = 0.$$

Suppose that the current $|i|$ flows along the closed curve C_0 . From (34) it will be seen that $H_x/4\pi|i|$ becomes infinite at C_0 after the manner of the potential due to a distribution of doublets or dipoles along C_0 and with axes parallel to the plane $x=0$. Applying the results of §5 to the reflection of H_x across $x=0$ (the constant a is now equal to α/μ and satisfies (9)), one is led (so far as H_x is concerned) to an image of the original current along C_1 , the reflection of C_0 in $x=0$, as well as to a current sheet or solenoid along the cylinder which projects C_0, C_1 on the plane $x=0$, the current density along the latter being $-2(\alpha/\mu)e^{\alpha\mu(x+h)}$ per unit x for $x < -h$. The remaining field components may be shown likewise to agree with the field of the original current and the images just described.

The distributed currents, it will be noted, are not "in phase" with the original current.

If C_0 is parallel to the plane $x=0$, then the electric field is likewise parallel to it, and the boundary condition (37) is satisfied by its components E_y, E_z .

It is possible to express the induced electromotive in terms of

the mutual inductances between C_0 and its images in planes $x = \text{constant}$. It is intended to treat this subject at greater length at a future date and to present an exact treatment, including the nature of the reflections when (31) is not approximated by (35).

8. *Higher Normal Derivatives.* In this concluding section we shall consider the extension obtained when (4) is replaced by a boundary condition involving higher derivatives:

$$(38) \quad \frac{\partial^2 u}{\partial x^2} + a \frac{\partial u}{\partial x} + bu = 0 \quad \text{along } x = 0,$$

where a and b are constants; a proper restriction on a and b will be indicated below. It will be shown that a singularity at P_0 corresponding to a point charge will be reflected into two exponential trails along with a point charge.

Perhaps the most elegant manner of arriving at this conclusion is by means of the algorithm of Heaviside's operational calculus.*

Denote the charge density of u along the x -axis by $\rho(x)$ and replace $\partial/\partial x$ by p . At P_0 the charge density of u would be denoted by $S_0(x-h)$, where the singular function $S_0(x)$ represents the "unit impulse" at $x=0$. Applying the left-hand operator of (38) to u , one obtains a harmonic function which vanishes at $x=0$. The charge distribution of this function is given by $(p^2 + ap + b)\rho(x)$, and for $x > 0$ this reduces to $(p^2 + ap + b) \cdot S_0(x-h)$. The negative reflection in $x=0$ leads to the further charge $-(p^2 - ap + b)S_0(x+h)$. Hence

$$(39) \quad \begin{aligned} (p^2 + ap + b)\rho(x) &= (p^2 + ap + b)S_0(x-h) \\ &\quad - (p^2 - ap + b)S_0(x+h). \end{aligned}$$

Solving for $\rho(x)$, we obtain

$$(40) \quad \begin{aligned} \rho(x) &= S_0(x-h) - \left(\frac{p^2 - ap + b}{p^2 + ap + b} \right) S_0(x+h) \\ &= S_0(x-h) - \left(1 + \frac{A}{p-\alpha} + \frac{B}{p-\beta} \right) S_0(x+h). \end{aligned}$$

* The reader who is unfamiliar with the latter will prefer to regard the following not as a proof but as a heuristic lead toward a result which is to be substantiated in some other way, for instance, by means of Fourier or Bromwich integrals, or by direct substitution in (38) of the potential derived.

To obtain the last equation the partial fraction resolution

$$(41) \quad \frac{p^2 - ap + b}{p^2 + ap + b} = 1 - \frac{2ap}{p^2 + ap + b} = 1 - \left(\frac{A}{p - \alpha} + \frac{B}{p - \beta} \right)$$

was used; here α, β are the roots of $p^2 + ap + b = 0$. Interpreting (40) for the case where $R(\alpha) > 0, R(\beta) > 0$, one obtains

$$(42) \quad \rho(x) = S_0(x - h) - S_0(x + h) - [Ae^{\alpha(x+h)} + Be^{\alpha(x+h)}]H(-x - h),^*$$

where $H(x)$ is unity for positive x and zero for negative x . Thus there is a negative point charge at $x = -h$ and a distributed charge of density $-[Ae^{\alpha(x+h)} + Be^{\alpha(x+h)}]$ for $x < -h$. Since $A\beta + B\alpha = 0$, the total amount of distributed charge vanishes.

It is of interest to point out that the content of (10), (11), and (22) can be derived in a similar fashion. In the former case one condenses the argument of §3 into

$$(43) \quad (p - \alpha)\rho(x) = (p - a)S_0(x - h) + (p + a)S_0(x + h);$$

in the latter case, the charge density at P_0 is $pS_0(x - h)$, and one obtains, by applying $(p - a)$ and using negative reflection in $x = 0$,

$$(44) \quad (p - a)\rho(x) = (p - a)pS_0(x - h) - (p + a)pS_0(x + h).$$

Solving (43), (44) for $\rho(x)$ and using partial fractions, one obtains

$$(45) \quad \rho(x) = S_0(x - h) + S_0(x + h) + 2a/(p - a)S_0(x + h),$$

$$(46) \quad \rho(x) = pS_0(x - h) - [2a + p + 2a^2/(p - a)]S_0(x + h),$$

respectively, and, interpreting $[1/(p - a)]S_0(x + h)$ as in (40), one is led to the charges implied in (10), (23).

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* The present interpretation of $[1/(p - \alpha)]S_0(x + h)$ differs from that of the orthodox operational calculus and corresponds to using a Bromwich integral with the path of integration lying to the left of the origin. This could have been avoided by placing the original singularity along the negative x -axis.

The more customary notation is 1 in place of $H(x)$ and $p1$ in place of $S_0(x)$.