# SOME SYMBOLIC IDENTITIES*

BY L. I. NEIKIRK

Differential equations were first solved by symbolic methods in England and on the continent in the first half of the last century. The differentiation symbol was treated as a symbol of quantity with restrictions. Then followed symbolic treatment of invariants and covariants, Cayley's hyperdeterminant, and Aronhold's symbolic notations. These were followed by Blissard's umbral notation in the theory of numbers.

This paper is devoted to showing that these are all reducible to symbolic differentiation.

If we represent differentiation by the symbol $D$ and separate the symbols of operation from the symbols of quantity, then any analytic identity, such as $\Phi_1(y)=\Phi_2(y)$, will give an operational identity, $\Phi_1(D)=\Phi_2(D)$.

If this operational identity is applied to a second identity

$$F_1(x) = F_2(x)$$

the result will be a new identity. Most identities obtained in this way are easily obtained otherwise. The following are some examples.

*Invariants and covariants.* If $D_1=\partial/\partial x_1$ and $D_2=\partial/\partial x_2$, then

$$D_2^r D_1^{n-r}(\alpha_1 x_1 + \alpha_2 x_2)^n = n!\alpha_1^{n-r}\alpha_2^r,$$

where $\alpha_x^n$ is a special form of degree $n$, while the operation on the general form gives

$$D_2^r D_1^{n-r} f(x_1, x_2) = D_2^r D_1^{n-r}(a_0 x_1^n + n a_1 x_1^{n-1} x_2 + \cdots) = n! a_r.$$

We now transform our coordinates:

$$x_1 = \xi_1 X_1 + \eta_1 X_2, \qquad x_2 = \xi_2 X_1 + \eta_2 X_2,$$

or

$$X_1 = \frac{1}{\Delta}(\eta_2 x_1 - \eta_1 x_2), \qquad X_2 = \frac{1}{\Delta}(-\xi_2 x_1 + \xi_1 x_2),$$

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where

$$\Delta = \begin{vmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{vmatrix} \neq 0.$$

Then if $\partial/\partial X_1 = \overline{D}_1$ and $\partial/\partial X_2 = \overline{D}_2$, we write

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial X_1} \cdot \frac{\partial X_1}{\partial x_1} + \frac{\partial f}{\partial X_2} \frac{\partial X_2}{\partial x_1}$$

in symbolic form

$$D_1 f = \frac{1}{\Delta} (\overline{D}_1 \eta_2 - \overline{D}_2 \xi_2) f,$$

also

$$D_2 f = \frac{1}{\Delta} (- \overline{D}_1 \eta_1 + \overline{D}_2 \xi_2) f;$$

and if $D' = \partial/\partial y$ we have

$$\begin{aligned} (DD') &= (D_1 D_2' - D_2 D_1') \\ &= \frac{1}{\Delta^2} [(\overline{D}_1 \eta_2 - \overline{D}_2 \xi_2)(- \overline{D}_1' \eta_1 + \overline{D}_2' \xi_1) \\ &\qquad - (- \overline{D}_1 \eta_1 + \overline{D}_2 \xi_1)(\overline{D}_1' \eta_2 - \overline{D}_2' \xi_2)] \\ &= \frac{1}{\Delta^2} \begin{vmatrix} \eta_2 & - \xi_2 \\ - \eta_1 & \xi_1 \end{vmatrix} \begin{vmatrix} \overline{D}_1 & \overline{D}_1' \\ \overline{D}_2 & \overline{D}_2' \end{vmatrix} = \frac{1}{\Delta} \begin{vmatrix} \overline{D}_1 & \overline{D}_1' \\ \overline{D}_2 & \overline{D}_2' \end{vmatrix}. \end{aligned}$$

For example

$$f(x)_a = a_0 x_1^4 + 4a_1 x_1^3 x_2 + 6a_2 x_1^2 x_2^2 + 4a_3 x_1 x_2^3 + a_4 x_2^4$$

gives

$$\frac{1}{2} \begin{vmatrix} D_1 & D_1' \\ D_2 & D_2' \end{vmatrix}^4 f(x)_a f(y)_b \Big]_{\substack{b=a \\ y=x}} = (4!)^2 (a_0 a_4 - 4a_1 a_3 + 3a_2^2),$$

while

$$\frac{1}{2} \begin{vmatrix} \overline{D}_1 & \overline{D}_1' \\ \overline{D}_2 & \overline{D}_2' \end{vmatrix}^4 F(X)_A F(Y)_B \Big]_{\substack{B=A \\ Y=X}} = (4!)^2 (A_0 A_4 - 4A_1 A_3 + 3A_2^2).$$

Therefore

$$A_0 A_4 - 4A_1 A_3 + 3A_2^2 = \Delta^4 (a_0 a_4 - 4a_1 a_3 + 3a_2^2).$$

This is Cayley's hyperdeterminant notation. This would be, in the Aronhold symbolic notation,

$$\frac{1}{2}\begin{vmatrix} D_1 & D_1' \\ \Delta_2 & \Delta_2' \end{vmatrix}^4 \alpha_x{}^4\beta_y{}^4\Big]_{y=x} = \frac{1}{2}(4!)^2(\alpha\beta)^4.$$

*Blissard's umbral notation.* Let $D = d/dy$. Then the symbolic form of MacLaurin's theorem is

$$F(x) = F(y)\Big]_{y=0} + DF(y)\Big]_{y=0} x + \frac{D^2}{1.2}F(y)\Big]_{y=0} x^2 + \cdots,$$

or

$$F(x) = \left(1 + xD + \frac{x^2D^2}{z!} + \cdots\right)F(y)\Big]_{y=0}.$$

Now if

$$F(y) = 1 + B_1y + \frac{B_2y^2}{2!} + \frac{B_3y^3}{3!} + \cdots, \quad (e^{By}),$$

then

$$F(y)\Big]_{y=0} = 1;\ DF(y)\Big]_{y=0} = B_1;\ D^2F(y)\Big]_{y=0} = B_2; \cdots;$$

and

$$\left(1_1 + nB_1 + \frac{n(n-1)}{1\cdot 2}B_2 + \cdots + nB_{n-1} + B_n\right) - B_n$$
$$= [(1 + B)^n - B^n] = 0$$

becomes

$$\left\{\left[\left(1 + nD + \frac{n(n-1)}{2!}D^2 + \cdots + nD^{n-1} + D^n\right) - D^n\right]F(y)\right\}_{y=0}$$
$$= \{[(1 + D)^n - D^n]F(y)\}_{y=0} = 0;$$

and one of the principal identities used in Blissard's theory becomes

$$\{[f(x + (D - 1)\theta) - f(x - \theta D)]F(y)\}_{y=0} = \theta \frac{df(x)}{dx} \cdot$$

Blissard's remark, "*An equation which has a representative quantity is not susceptible to any algebraic operation by which the indices would be affected,*" becomes

$$(Df)^2 \neq D^2 f.$$

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# ON FOURTH ORDER SELF-ADJOINT DIFFERENCE SYSTEMS*

BY V. V. LATSHAW

A linear difference expression for which the differential transform is self-adjoint (anti-self-adjoint) we shall call self-adjoint (anti-self-adjoint).† We choose two fourth order difference equations

$$(1) \quad \begin{aligned} L^+(u) \equiv\ & p(x)[u(x + 2) + u(x - 2)] \\ & + \lambda[u(x + 1) + u(x - 1)] + R(x)u(x) = 0, \end{aligned}$$

$$(2) \quad \begin{aligned} L^-(u) \equiv\ & p(x)[u(x + 2) - u(x - 2)] \\ & + \lambda[u(x + 1) - u(x - 1)] = 0, \end{aligned}$$

where $L^+(u)$ is self-adjoint and $L^-(u)$ anti-self-adjoint for the range $(x=a, a+1, \cdots, b-1; b-a \geqq 4)$. $R(x)$ and $p(x)$ are both real, $p(x)$ being a non-vanishing periodic function of period two; $\lambda$ is a parameter.

Let the functions $(y_1, y_2, y_3, y_4)$ constitute a fundamental set of solutions for either (1) or (2), and $(w_1, w_2, w_3, w_4)$ the set adjoint to it. The two sets are related by the equations

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† J. Kaucky, *Sur les équations aux différences finies qui sont identiques à leurs adjointes*, Publications of the Faculty of Sciences, University of Masaryk, No. 22 (1922). For a discussion of adjoint differential expressions of infinite order, see H. T. Davis, *The Theory of Linear Operators*, 1936, pp. 474–475.