ON THE REMAINDER IN THE APPROXIMATE EVALUATION OF THE PROBABILITY IN THE SYMMETRICAL CASE OF JAMES BERNOULLI'S THEOREM*

BY C. D. OLDS

1. *Introduction.* In this paper we consider the symmetrical case of James Bernoulli's theorem in the theory of probability. We let m represent the number of successes of an event in a series of n independent trials with constant probability $p=1-q=1/2$ for the success of each trial. Then we seek the probability P of the inequality

$$\left| m - \frac{n}{2} \right| \leqq \epsilon, \tag{1}$$

where ϵ is a given arbitrary positive number. The probability P is usually given by an approximate formula without mention of the error term or remainder involved.† In 1926, D. Mirimanoff‡ discussed this error term and gave results which are similar, but not as free from restrictions as those obtained here by entirely different methods.§

2. *The Exact Expression for P.* Let T_m represent the probability of m successes in the n trials and consider its generating function

$$\sum_{m=0}^{n} T_m t^m,$$

where t is an arbitrary variable. It has been shown|| that

---

* Presented to the Society, April 3, 1937.

† See, for example, I. Todhunter, *A History of the Mathematical Theory of Probability*, 1865, pp. 548–553.

‡ D. Mirimanoff, *Le jeu de pile ou face et les formules de Laplace et de J. Eggenberger*, Commentarii Mathematici Helvetici, vol. 2 (1926), pp. 133–168.

§ The author wishes to acknowledge the assistance rendered him by Professor J. V. Uspensky.

|| For this and similar results see A. A. Markoff, *Wahrscheinlichkeitsrechnung*, 1912, pp. 18–44.

$$f(t) = \sum_{m=0}^{n} T_m t^m = (\tfrac{1}{2})^n (t+1)^n.$$

In this last expression we set $t = e^{i\phi}$, multiply through by $e^{-im\phi}$, and then integrate between the limits $-\pi$ and π; we find

$$T_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})e^{-im\phi}d\phi,$$

since

$$\int_{-\pi}^{\pi} e^{(m-n)i\phi}d\phi = \begin{cases} 0, & m \neq n, \\ 2\pi, & m = n. \end{cases}$$

Now let $\epsilon = -\frac{1}{2} + (n/4)^{1/2}\zeta$ and express the inequality (1) in the form

$$l_1 \leqq m \leqq l_2,$$

where l_1 and l_2 are integers. Then the probability P has the exact expression

$$P = \sum_{m=l_1}^{l_2} T_m = \sum_{m=l_1}^{l_2} \frac{1}{2\pi}\int_{-\pi}^{\pi} (\tfrac{1}{2})^n (e^{i\phi}+1)^n e^{-im\phi}d\phi$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\cos\frac{\phi}{2}\right)^n e^{in\phi/2}\sum_{m=l_1}^{l_2} e^{-im\phi}d\phi.$$

Using the known identity

$$\sum_{m=l_1}^{l_2} e^{-im\phi} = \frac{i}{2\sin\dfrac{\phi}{2}}\left\{e^{-i(l_2+1/2)\phi} - e^{-i(l_1-1/2)\phi}\right\},$$

and substituting the values of l_1 and l_2, we find that

$$P = \frac{1}{\pi}\int_0^{\pi}\left(\cos\frac{\phi}{2}\right)^n \frac{\sin(\frac{1}{2}\zeta n^{1/2}\phi)}{\sin\dfrac{\phi}{2}}d\phi. \tag{2}$$

3. *Three Lemmas.* Let λ be an arbitrary number such that $0 < \lambda < \pi$. We use the expansion*

* L. L. Smail, *Elements of the Theory of Infinite Processes*, 1923, p. 245.

$$-\log\cos x=(2^2-1)\frac{2}{2!}B_1x^2+\frac{1}{2}(2^4-1)\frac{2^3}{4!}B_2x^4$$
$$+\frac{1}{3}(2^6-1)\frac{2^5}{6!}B_3x^6+\cdots,$$

where $B_1, B_2, B_3, \cdots$ are the Bernoullian numbers. Consequently all the coefficients in this expansion are positive. Hence, we can deduce

$$-\log\cos\frac{\phi}{2}=\frac{\phi^2}{8}+M\phi^4, \tag{3}$$

$$-\log\cos\frac{\phi}{2}=\frac{\phi^2}{8}+\frac{\phi^4}{192}+N\phi^6, \tag{4}$$

where

$$0<M\leqq\lambda^{-4}\left(\log\sec\frac{\lambda}{2}-\frac{\lambda^2}{8}\right)=a,$$

$$0<N\leqq\lambda^{-6}\left(\log\sec\frac{\lambda}{2}-\frac{\lambda^2}{8}-\frac{\lambda^4}{192}\right)=b,$$

provided $0<\phi\leqq\lambda$.

Likewise, from the expansion*

$$\frac{\phi}{2\sin\frac{\phi}{2}}=1+\sum_{k=1}^{\infty}\frac{(2^{2k}-2)}{(2k)!}B_k\left(\frac{\phi}{2}\right)^{2k},$$

where again all the coefficients are positive, we find that

$$\frac{1}{\sin\frac{\phi}{2}}=\frac{2}{\phi}+\frac{\phi}{12}+L\phi^3, \tag{5}$$

where

$$0<L\leqq\lambda^{-3}\left(\csc\frac{\lambda}{2}-\frac{2}{\lambda}-\frac{\lambda}{12}\right)=c,$$

provided $0<\phi\leqq\lambda$.

---

* K. Knopp, *Theory and Application of Infinite Series*, 1928, p. 204.

Using (3) we can easily show that

$$(6)\qquad \left|\left(\cos\frac{\phi}{2}\right)^n - e^{-(n\phi^2)/8}\right| < na\phi^4 e^{-(n\phi^2)/8}, \qquad 0 < \phi \leqq \lambda.$$

From (4) we find that

$$\left(\cos\frac{\phi}{2}\right)^n - e^{-(n\phi^2)/8}\left(1 - \frac{n\phi^2}{192}\right) = (-nN\phi^6 + \tfrac{1}{2}\xi n^2 M^2\phi^8)e^{-(n\phi^2)/8},$$

where $0<\xi<1$, and consequently

$$(7)\qquad \left|\left(\cos\frac{\phi}{2}\right)^n - e^{-(n\phi^2)/8}\left(1 - \frac{n\phi^2}{192}\right)\right| < (nb\phi^6 + \tfrac{1}{2}n^2a^2\phi^8)e^{-(n\phi^2)/8}, \qquad 0 < \phi \leqq \lambda.$$

4. *Application of* (5), (6), *and* (7) *to* (2). Applying (5) to the integral on the right of (2), we have

$$(8)\qquad \begin{aligned} P = {} & \frac{2}{\pi}\int_0^\lambda \left(\cos\frac{\phi}{2}\right)^n \frac{\sin(\frac{1}{2}\zeta n^{1/2}\phi)}{\phi}\,d\phi \\ & + \frac{1}{12\pi}\int_0^\lambda \left(\cos\frac{\phi}{2}\right)^n \cdot\phi\cdot\sin(\tfrac{1}{2}\zeta n^{1/2}\phi)\,d\phi \\ & + \frac{1}{\pi}\int_0^\lambda \left(\cos\frac{\phi}{2}\right)^n \cdot L\cdot\phi^3\cdot\sin(\tfrac{1}{2}\zeta n^{1/2}\phi)\,d\phi \\ & + \frac{1}{\pi}\int_\lambda^\pi \left(\cos\frac{\phi}{2}\right)^n \cdot\frac{\sin(\frac{1}{2}\zeta n^{1/2}\phi)}{\sin\frac{\phi}{2}}\,d\phi. \end{aligned}$$

For brevity, we shall let the integrals in (8) be denoted by I_1, I_2, I_3, and I_4 respectively.

The inequality (7) shows that

$$(9)\qquad I_1 = \frac{2}{\pi}\int_0^\lambda e^{-(n\phi^2)/8}\left(1 - \frac{n\phi^4}{192}\right)\frac{\sin(\frac{1}{2}\zeta n^{1/2}\phi)}{\phi}\,d\phi + \Delta_{(1)},$$

where

$$|\Delta_{(1)}| < \frac{2}{\pi}\int_0^\infty \left(nb\phi^5 + \frac{n^2a^2}{2}\phi^7\right)e^{-(n\phi^2)/8}d\phi = \frac{1024b + 12288a^2}{\pi n^2}.$$

The integral in (9) splits into two integrals, the first of which gives

$$\frac{2}{\pi}\int_0^\lambda e^{-(n\phi^2)/8}\frac{\sin\left(\frac{1}{2}\zeta n^{1/2}\phi\right)}{\phi}d\phi \tag{10}$$
$$= \frac{2}{\pi}\int_0^\infty e^{-(n\phi^2)/8}\frac{\sin\left(\frac{1}{2}\zeta n^{1/2}\phi\right)}{\phi}d\phi + \Delta_{(2)},$$

where

$$|\Delta_{(2)}| < \frac{2}{\pi}\int_\lambda^\infty e^{-(n\phi^2)/8}\frac{d\phi}{\phi} < \frac{8}{\pi\lambda^2 n}e^{-\lambda^2 n/8},$$

since

$$\int_x^\infty e^{-u^2}\frac{du}{u} < \frac{e^{-x^2}}{2x^2}, \qquad x > 0.$$

The integral in the right member of (10) is

$$\frac{2}{\pi}\int_0^\infty e^{-(n\phi^2)/8}\frac{\sin\left(\frac{1}{2}\zeta n^{1/2}\phi\right)}{\phi}d\phi = \frac{2}{\pi}\int_0^\infty e^{-(v^2)/2}\frac{\sin(\zeta v)}{v}dv$$
$$= \left(\frac{2}{\pi}\right)^{1/2}\int_0^\zeta e^{-(v^2)/2}dv.$$

We replace the second integral from (9) by

$$\frac{2n}{192\pi}\int_0^\infty e^{-(n\phi^2)/8}\phi^3\sin\left(\tfrac{1}{2}\zeta n^{1/2}\phi\right)d\phi + \Delta_{(3)},$$

where

$$|\Delta_{(3)}| < \frac{2n}{192\pi}\int_\lambda^\infty e^{-(n\phi^2)/8}\phi^3 d\phi = \frac{1}{3\pi n}\left(1 + \frac{n\lambda^2}{8}\right)e^{-(n\lambda^2)/8}.$$

We have

$$\frac{2n}{192\pi}\int_0^\infty e^{-(n\phi^2)/8}\phi^3\sin\left(\tfrac{1}{2}\zeta n^{1/2}\phi\right)d\phi = \frac{3\zeta - \zeta^3}{6n(2\pi)^{1/2}}e^{-(\zeta^2)/2},$$

as is clear if we differentiate three times with respect to α the integral

$$(11)\qquad \int_0^\infty e^{-\alpha u^2}\cos(\beta u)du = \frac{1}{2}\left(\frac{\pi}{\alpha}\right)^{1/2} e^{-\beta/(4\alpha)}, \qquad \alpha > 0,$$

and make obvious substitutions.

5. *The Integral* I_2. If we apply the inequality (6) to I_2 we get

$$I_2 = \frac{1}{12\pi}\int_0^\lambda e^{-(n\phi^2)/8}\phi \sin\left(\tfrac{1}{2}\zeta n^{1/2}\phi\right)d\phi + \Delta_{(4)},$$

where

$$\left|\Delta_{(4)}\right| < \frac{na}{12\pi}\int_0^\infty e^{-(n\phi^2)/8}\phi^5\cdot d\phi = \frac{128a}{3\pi n^2}\cdot$$

Also,

$$\frac{1}{12\pi}\int_0^\lambda e^{-(n\phi^2)/8}\phi \sin\left(\tfrac{1}{2}\zeta n^{1/2}\phi\right)d\phi$$

$$= \frac{1}{12\pi}\int_0^\infty e^{-(n\phi^2)/8}\phi \sin\left(\tfrac{1}{2}\zeta n^{1/2}\phi\right)d\phi + \Delta_{(5)},$$

where

$$\left|\Delta_{(5)}\right| < \frac{1}{12\pi}\int_\lambda^\infty e^{-(n\phi^2)/8}\phi d\phi = \frac{1}{3\pi n}e^{-(n\lambda^2)/8}.$$

Using (11) again we find that

$$\frac{1}{12\pi}\int_0^\infty e^{-(n\phi^2)/8}\phi \sin\left(\tfrac{1}{2}\zeta n^{1/2}\phi\right)d\phi = \frac{2\zeta}{6n(2\pi)^{1/2}}e^{-(\zeta^2)/2}.$$

6. *The Integrals* I_3 *and* I_4. For the integral I_3 we have

$$\left|I_3\right| \leqq \frac{1}{\pi}\int_0^\lambda\left(\cos\frac{\phi}{2}\right)^n\phi^3\cdot L\cdot d\phi < \frac{c}{\pi}\int_0^\infty e^{-(n\phi^2)/8}\phi^3 d\phi = \frac{32c}{\pi n^2}\cdot$$

Likewise for I_4 we have

$$\left|I_4\right| \leqq \frac{1}{\pi}\int_\lambda^\pi\left(\cos\frac{\phi}{2}\right)^n\cdot\frac{d\phi}{\sin\dfrac{\phi}{2}} \leqq \int_\lambda^\pi\left(\cos\frac{\phi}{2}\right)^n\frac{d\phi}{\phi},$$

since, for $\phi \leqq \pi$,

$$\sin\frac{\phi}{2} \geqq \frac{\phi}{\pi}$$

Also,

$$\int_{\lambda}^{\pi}\left(\cos\frac{\phi}{2}\right)^{n}\frac{d\phi}{\phi} < \int_{\lambda}^{\infty} e^{-(n\phi^2)/8}\frac{d\phi}{\phi} < \frac{4}{n\lambda^2}e^{-(n\lambda^2)/8},$$

which shows that

$$|I_4| < \frac{4}{n\lambda^2}e^{-(n\lambda^2)/8}.$$

7. *The Remainder* Δ. *Conclusion.* Combining the above results we find that the probability P of the inequality

$$\left| m - \frac{n}{2} \right| \leqq -\frac{1}{2} + \zeta\left(\frac{n}{4}\right)^{1/2}$$

is given by

$$P = \left(\frac{2}{\pi}\right)^{1/2}\int_0^{\zeta} e^{-(v^2)/2}dv + \frac{\zeta^3 - \zeta}{6n(2\pi)^{1/2}}\cdot e^{-(\zeta^2)/2} + \Delta,$$

where for the remainder or error term Δ we have

$$|\Delta| < \left[\frac{8}{n\lambda^2\pi} + \frac{4}{n\lambda^2} + \frac{1}{3n\pi}(2 + \tfrac{1}{8}n\lambda^2)\right]e^{-(n\lambda^2)/8} + \frac{\omega}{n^2\pi},$$

with

$$\omega = \frac{128}{3}a + 12288a^2 + 1024b + 32c.$$

We now let λ take on numerical values between 1 and 2, and seek, corresponding to each, the smallest value of n such that $|\Delta| < (n^{-2})/2$. It is found that, by taking $\lambda = 1.8$, we have the inequality

$$|\Delta| < \frac{1}{2n^2}$$

for $n \geqq 17$.

STANFORD UNIVERSITY