$$\left| \int_\alpha^\beta d[f(x)g(x)] - \int_\alpha^\beta g(x)df(x) - \int_\alpha^\beta f(x)dg(x) \right|$$

is less than ϵ multiplied by the total variation of $f(x)$ plus the total variation of $g(x)$. Hence this expression must be equal to 0 and therefore the formula for integration by parts is valid under the above hypotheses.

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A SET OF POSTULATES FOR BOOLEAN ALGEBRA

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1. *A New Set of Postulates*. In the development of a Boolean Algebra, Boole's Law of Development

$$f(x) = f(1)x + f(0)x',$$

stands out as a basic relationship. This law is so all embracing that the question naturally arises, if this is set as a postulate, what postulates in addition to it are needed to define a Boolean Algebra? Using as undefined a class K and the Sheffer stroke function, we shall show that, in addition to a form of Boole's Law, only two "trivial" postulates are required.

POSTULATES.*

I. *K contains at least two elements.*

II. *If a and b are elements of K, then a/b is an element of K.*

Definitions: $a' = a/a$, $a \cdot b = a'/b'$, and $a + b = (a/b)'$.

III. *There exists in K a unique element* 0, *such that, if $f(x)$ is any function definable in terms of / and elements of K, we have, for any x in K,*

$$f(x) = f(0')x + f(0)x'.$$

THEOREM 1. $0'' = 0$.

Proof: From III, and the preceding definitions, we have

$$(1) \qquad x = 0'x + 0x' = [(0'x)/(0x')]';$$

in particular

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* This is the smallest set of postulates for a Boolean Algebra yet given.

$$0 = [(0'0)/(00')]'.$$

Thus there exists in K an element 1, such that

$$0 = 1'.$$

From (1) above

$$1 = 0'1 + 01.,$$

and

$$1 = 1''1 + 1'1'.$$

Letting $f(x) = x''x + x'x'$ in III, we have

$$x''x + x'x' = (0'''0' + 0''0'')x + (0''0 + 0'0')x';$$

in particular

$$1 = 1''1 + 1'1' = (0'''0' + 0''0'')1 + (0''0 + 0'0')1'.$$

This becomes

$$1 = 0'''1 + 0'1',$$

since letting $f(x)$ equal x'' and x' respectively in III yields

$$x'' = 0'''x + 0''x', \quad \text{and} \quad x' = 0''x + 0'x';$$

in particular

$$0''' = 0'''0' + 0''0'', \quad \text{and} \quad 0' = 0''0 + 0'0'.$$

Letting $f(x) = 0'''x + x''x'$ in III, we have

$$0'''x + x''x' = (0'''0' + 0'''0'')x + (0'''0 + 0''0')x';$$

in particular

$$1 = 0'''1 + 1''1' = (0'''0' + 0'''0'')1 + (0'''0 + 0''0')1'.$$

This becomes

$$1 = 0'''1 + 0''1',$$

since

$$0''' = 0'''0' + 0'''0'', \quad \text{and} \quad 0'' = 0'''0 + 0''0'.$$

From III, $x'' = 0'''x + 0''x'$; thus $1'' = 0'''1 + 0''1'$, and $1 = 1''$. Hence $1' = 1'''$ and $0 = 0''$.

THEOREM 2. $x'' = x$.

Proof: $x''=0'''x+0''x'=0'x+0x'=x$.

From Theorem 1 and the definition of 1, we have $0'=1$, and thus

$$0/0 = 0' = 1, \quad \text{and} \quad 1/1 = 1' = 0.$$

THEOREM 3. $1/0=0/1=1$.

Proof: From III and the definitions, we have

$$1 = 0'1 + 01' = 0/0 + 1/1 = 1 + 0 = (1/0)';$$
$$0 = 1/0.$$
$$0' = 0''0 + 0'0' = 1/1 + 0/0 = 0 + 1 = (0/1)';$$
$$0 = 0/1.$$

THEOREM 4. $0+0=0$, $1+0=1$, $0+1=1$, $1+1=1$, $00=0$, $10=0$, $01=0$, and $11=1$.

Proof: These equations follow immediately upon using the results of the preceding theorems in the definitions of $+$ and $\cdot$.

In the following theorems the equations are obtained by letting $f(x)$ equal the left-hand side.

THEOREM 5. $1x=x=x\,1=0+x=x+0$.

Proof:
$$1x=(1\ 1)x+(1\ 0)x'=1x+0x',$$
$$x=1x+0x',$$
$$x1=(1\ 1)x+(0\ 1)x'=1x+0x',$$
$$0+x=(0+1)x+(0+0)x'=1x+0x',$$
$$x+0=(1+0)x+(0+0)x'=1x+0x';$$

since all five are equal to $1x+0x'$, the theorem follows.

THEOREM 6. $0x=x0=0=xx'$.

Proof:
$$0x=(0\ 1)x+(0\ 0)x'=0x+0x',$$
$$x0=(1\ 0)x+(0\ 0)x'=0x+0x',$$
$$0=0x+0x',$$
$$xx'=(1\ 1')x+(0\ 0')x'=0x+0x';$$

since all four are equal to $0x+0x'$, the theorem follows.

THEOREM 7. $1+x=x+1=1=x+x'$.

Proof:
$$1+x=(1+1)x+(1+0)x'=1x+1x',$$
$$x+1=(1+1)x+(0+1)x'=1x+1x',$$
$$1=1x+1x',$$
$$x+x'=(1+1')x+(0+0')x'=1x+1x';$$

since all four are equal to $1x+1x'$, the theorem follows.

THEOREM 8. $ax=xa$.

Proof: $$ax=(a\ 1)x+(a\ 0)x'=(1\ a)x+(0\ a)x'=xa.$$

THEOREM 9. $a+x=x+a$.

Proof: $$a+x=(a+1)x+(a+0)x'=(1+a)x+(0+a)x'=x+a.$$

THEOREM 10. $x+bc=(x+b)(x+c)$.

Proof: $$\begin{aligned}(x+b)(x+c)&=(1+b)(1+c)x+(0+b)(0+c)x'\\&=(1)(1)x+(b)(c)x'=1x+bc\ x'\\&=(1+bc)x+(0+bc)x'=x+bc.\end{aligned}$$

THEOREM 11. $xb+xc=x(b+c)$.

Proof: $$\begin{aligned}xb+xc&=(1b+1c)x+(0b+0c)x'=(b+c)x+0x'\\&=(b+c)x=x(b+c).\end{aligned}$$

The postulates we have given are known to be true in a Boolean Algebra, therefore they are necessary. We shall show that they are sufficient by showing that Huntington's postulates are derivable from them.

2. *Huntington's Postulates and their Derivation.* The following is Huntington's set of postulates; to each is appended a brief indication of its derivation from those of our set.

1. (a) *If a and b are elements of K, then $a+b$ is an element of K.* By definition, $a+b=(a/b)'=(a/b)/(a/b)$; by Postulate I, if a and b are elements of K, a/b is an element, and $a+b$ is an element.

(b) *If a and b are elements of K, then $a\cdot b$ is an element of K.* By definition, $a\cdot b=a'/b'=(a/a)/(b/b)$ is an element of K as above.

2. (a) *There exists an element 0 in K such that $a+0=a$.*

(b) *There exists an element 1 in K such that $a\cdot 1=a$.*
Theorem 5.

3. (a) *If a, b, $a+b$, $b+a$ belong to K, then $a+b=b+a$.*
Theorem 9.

(b) *If a, b, ab, ba belong to K, then $ab=ba$.*
Theorem 8.

4. (a) *If a, b, c, bc, $a+bc$, $a+b$, $a+c$, and $(a+b)(a+c)$ belong to K, then $a+bc=(a+b)(a+c)$.*
Theorem 10.

(b) *If* $a, b, c, b+c, a(b+c), ab, ac$, *and* $ab+ac$ *belong to* K, *then* $ab+ac=a(b+c)$.
Theorem 11.

5. *If* 0 *and* 1 *exist and are unique, then for every element* a *belonging to* K *there exists an element* a' *in* K *such that* $a+a'=1$ *and* $aa'=0$.
Theorems 1, 6, and 7.

6. *There are at least two distinct elements in* K.
Postulate I.

3. *A Two Element Boolean Algebra.* A set of postulates for a two element Boolean Algebra can be obtained by changing Postulate I to: "K contains two and only two elements."

4. *Independence Examples*:

1. $K = \{\alpha\}$, $\alpha/\alpha = \alpha$.

2. $K = \{0, \beta\}$,

$'$	0	β	γ	δ
0	δ	γ	β	0
β	γ	γ	0	0
γ	β	0	β	0
δ	0	0	0	0

3. $K = \{\alpha, \beta\}$,

$'$	α	β
α	α	α
β	α	α

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