NOTE ON THE CONTINUITY OF THE ERGODIC FUNCTION

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1. *Introduction.* Let M denote a bounded point set and ϵ an arbitrarily chosen positive number. A continuous curve C is termed *ϵ-ergodic to M* if an arbitrary point of M lies at a distance $\leqq \epsilon$ from some point of C. Recently* I have shown that the set of continuous, rectifiable curves ϵ-ergodic to M contains a member whose length furnishes an absolute minimum for the lengths of the curves in the set. This member was called an *ergodic curve of M* and its length the *ergodic function $\Lambda(\epsilon)$ of M.* The function $\Lambda(\epsilon)$ is finite and non-negative, being equal to zero if and only if $\epsilon \geqq \rho$, where ρ is the radius of the smallest circular region containing M. In addition $\Lambda(\epsilon)$ was proved to be a monotone non-increasing function of ϵ which is always continuous on the right.

In this note it is shown that $\Lambda(\epsilon)$ is also continuous on the left (and is therefore continuous in the ordinary sense). In the original version of this paper I was able to prove this result only for a value $\epsilon_0(<\rho)$ of ϵ for which the set M had an ergodic curve which was an "ordinary curve" (a continuous curve which is either of class C' or else made up of a finite number of arcs of class C'). The general result announced above is made possible by Lemma 2 below for which I am indebted to Professor von Neumann.

2. *Preliminary Lemmas.* In this section we shall assemble a number of lemmas leading to the proof of the result announced in the introduction.

LEMMA 1. *The set M_1 of points lying at a distance $\leqq \epsilon$ from the points of a continuous rectifiable arc of length $2s$ joining two points A and B situated a distance $2c(c \leqq s)$ apart lies in a region composed of the points interior to two circles described about A and B as centers with radii equal to $\epsilon + (2^{1/2}\alpha + s/(2\epsilon))s$, where $\alpha^2 = 1 - c/s$.*

* *Ergodic curves*, American Journal of Mathematics, vol. 58 (1936), pp. 727–734.

For the proof of this lemma we first observe that a point on the rectifiable arc either lies in the interior of an ellipse E having A and B as foci and a major axis of length $2s$, or lies on E itself. The proof of this fact is elementary and is omitted.

The set E_ϵ of points lying at a distance $\leqq \epsilon$ from either the points of E or the points interior to E therefore contains the set M_1.

The boundary of E_ϵ is a closed analytic curve forming part of the envelope of the family of circles of radius ϵ with centers on E.

E_ϵ is divided into two halves by the minor axis of E produced. Let us consider that half which contains the focus A. An elementary calculation shows that an absolute maximum R of the distances of the points of this half of E_ϵ from A is furnished by the two points in which the minor axis of E produced intersects the boundary of E_ϵ. Upon computing this distance, we find

$$R = \left[\epsilon^2 + 2\epsilon(s^2 - c^2)^{1/2} + s^2\right]^{1/2}.$$

Since $c \leqq s$ and $s - c = \alpha^2 s$, it follows that

$$R \leqq \left[\epsilon^2 + 2^{3/2}\epsilon\alpha s + s^2\right]^{1/2} < \epsilon + \left(2^{1/2}\alpha + \frac{s}{2\epsilon}\right)s.$$

A circle of radius $\epsilon + (2^{1/2}\alpha + s/(2\epsilon))s$ described about A as a center accordingly contains in its interior the half of E_ϵ containing A. A similar circle described about B as a center will contain the remaining half of E_ϵ in its interior. The two circles taken together will then contain the whole of E_ϵ and will therefore *a fortiori* contain the set M_1 in their interiors.

LEMMA 2. *Let*

$$C\colon\ x = x(s), \quad y = y(s), \qquad 0 \leqq s \leqq L, \qquad (s = \text{arc length}),$$

be an open continuous rectifiable curve of length L. Let C be divided into n arcs of lengths $s_1^{(n)}, s_2^{(n)}, \cdots, s_n^{(n)}$ subtending chords of lengths $c_1^{(n)}, c_2^{(n)}, \cdots, c_n^{(n)}$ respectively. Define the n quantities $\alpha_1^{(n)}, \alpha_2^{(n)}, \cdots, \alpha_n^{(n)}$ by the equations

$$\alpha_i^{(n)} = \left[1 - \frac{c_i^{(n)}}{s_i^{(n)}}\right]^{1/2}, \qquad (i = 1, 2, \cdots, n),$$

and then the $n+1$ quantities $\bar{\alpha}_0^{(n)}, \bar{\alpha}_1^{(n)}, \cdots, \bar{\alpha}_n^{(n)}$ by the relations

$$\bar{\alpha}_0^{(n)} = \alpha_1^{(n)}, \quad \bar{\alpha}_i^{(n)} = \alpha_i^{(n)} + \alpha_{i+1}^{(n)} \quad for \quad i = 1, 2, \cdots, n-1,$$
$$\bar{\alpha}_n^{(n)} = \alpha_n^{(n)}.$$

If $s_1^{(n)} = s_2^{(n)} = \cdots = s_n^{(n)} = L/n$, *the arithmetic mean*

$$\bar{\bar{\alpha}}_n = \frac{1}{n+1} \sum_{i=0}^{n} \bar{\alpha}_i^{(n)}$$

of $\bar{\alpha}_0^{(n)}, \bar{\alpha}_1^{(n)}, \cdots, \bar{\alpha}_n^{(n)}$ *tends to zero as n tends to infinity.*

We have

$$\bar{\bar{\alpha}}_n = \frac{2}{n+1} \sum_{i=1}^{n} \alpha_i^{(n)} < \frac{2}{n} \sum_{i=1}^{n} \alpha_i^{(n)},$$

and therefore, upon substituting the values of $\alpha_i^{(n)}$ given above and applying Schwarz's inequality.

$$\bar{\bar{\alpha}}_n < \frac{2}{n} \left[n \sum_{i=1}^{n} \left(1 - \frac{c_i^{(n)}}{s_i^{(n)}} \right) \right]^{1/2},$$

whereupon, since $s_i^{(n)} = L/n$, it follows that

$$\bar{\bar{\alpha}}_n < 2 \left[1 - \frac{\sum_{i=1}^{n} c_i^{(n)}}{L} \right]^{1/2}.$$

Now C is rectifiable of length L. Therefore $\lim_{n \to \infty} \sum_{i=1}^{n} c_i^{(n)} = L$, and consequently $\lim_{n \to \infty} \bar{\bar{\alpha}}_n = 0$.

LEMMA 3. *Let*

$$C: \; x = x(s), \qquad y = y(s), \qquad 0 \leqq s \leqq L,$$

be a continuous rectifiable curve of length L. Denote by C_ϵ *the set of points lying at a distance* $\leqq \epsilon$ *from the points of C. There exists a sequence* $\{C_n\}$ *of continuous rectifiable curves for which, if* L_n *denotes the length of* C_n*, the following hold:*

(a) $\lim L_n = L$,

(b) C_n *is* ϵ_n*-ergodic to* C_ϵ *with* $\epsilon_n < \epsilon$ *and* $\lim \epsilon_n = \epsilon$.

In proving this lemma it is convenient to take C to be an open curve, the proof permitting an immediate adaptation to the case where C is closed. Divide C up into n equal arcs each of length L/n by inserting $n+1$ points on C. These $n+1$ points we shall denote by $0, 1, 2, \cdots, n$, where 0 and n denote the endpoints of C.

About a point i, $(i=0, 1, 2, \cdots, n)$, of the subdivision we describe two circles, one, Γ_i, of radius $\epsilon+(2^{1/2}\bar{\alpha}_i^{(n)}+L/(4n\epsilon))L/(2n)$, the other, γ_i, of radius $(2^{1/2}\bar{\alpha}_i^{(n)}+L/(4n\epsilon))L/n$, where in each case $\bar{\alpha}_i^{(n)}$ denotes the quantity defined in Lemma 2.

The totality of points interior to the circles Γ_i constitutes a point set which we designate by M_n. On placing $s=L/(2n)$ in Lemma 1, we ascertain that $C_\epsilon \subset M_n$.

A point making a complete circuit of a circle γ_i comes within a distance $\leqq(2^{1/2}\bar{\alpha}_i^{(n)}+L/(4n\epsilon))L/n$ of the points enclosed by γ_i, and within a distance $\leqq \epsilon-(2^{1/2}\bar{\alpha}_i^{(n)}+L/(4n\epsilon))L/(2n)$ of those points outside γ_i but inside Γ_i. For sufficiently great values of n the latter of the two distances is the greater. Consequently, for sufficiently great values of n, a point making a complete circuit of γ_i comes within a distance $\leqq \epsilon-(2^{1/2}\bar{\alpha}_i^{(n)}+L/(4n\epsilon))L/(2n) \leqq \epsilon - L^2/(8n^2\epsilon)$ of every point interior to Γ_i.

Now consider a point P which traverses C from one endpoint to the other and which makes complete circuits of all the circles γ_i during the process. More precisely, P moves as follows: starting at the endpoint 0 of C, the point P moves along C in the direction of increasing arc length until it meets the circle γ_0 with center 0 for the first time; P then makes a complete circuit of γ_0; after returning to the curve C, the point P continues along C in the direction of increasing arc length until it meets the circle γ_1 with center 1 for the first time; P then makes a complete circuit of γ_1; $\cdots$. At the completion of this process P coincides with n and will have traced out a continuous rectifiable curve C_n of length L_n, where

$$L_n = L + \sum_{i=0}^{n} 2\pi\left(2^{1/2}\bar{\alpha}_i^{(n)} + \frac{L}{4n\epsilon}\right)\frac{L}{n}.$$

On replacing $\sum_{i=0}^{n}\bar{\alpha}_i^{(n)}$ by its value $(n+1)\bar{\bar{\alpha}}_n$ (introduced in Lemma 2) and carrying out the remainder of the summation, we find

$$L_n = L\left[1 + 2\pi\frac{n+1}{n}\left(2^{1/2}\bar{\bar{\alpha}}_n + \frac{L}{4n\epsilon}\right)\right].$$

In addition the curve C_n is ϵ_n-ergodic to M_n, where

$$\epsilon_n = \epsilon - \frac{L^2}{8n^2\epsilon} < \epsilon.$$

The proof of the lemma is now completed by observing that, since $C_\epsilon \subset M_n$, the curve C_n is ϵ_n-ergodic to C_ϵ, and that

$$\lim \epsilon_n = \epsilon, \qquad \lim L_n = L,$$

the latter of which follows from Lemma 2.

3. *The Continuity of* $\Lambda(\epsilon)$. Let ϵ_0 denote a fixed value of ϵ lying in the interval $0<\epsilon<\rho$ (for the definition of ρ see the introduction). Let C be an ergodic curve of M for $\epsilon=\epsilon_0$. Write $\Lambda(\epsilon_0)=L$, so that the length of C is L. Let C_{ϵ_0} denote the set of points lying at a distance $\leqq \epsilon_0$ from points of C. We note that $M \subset C_{\epsilon_0}$. The ergodic function $\Lambda(\epsilon)$ will now be demonstrated to be continuous on the left for $\epsilon=\epsilon_0$.

If $\Lambda(\epsilon)$ be discontinuous on the left for $\epsilon=\epsilon_0$, we shall have, since $\Lambda(\epsilon)$ is monotone non-increasing

$$\lim \Lambda(\epsilon_n) = L + \delta, \qquad (\delta > 0),$$

for an arbitrarily selected sequence $\{\epsilon_n\}$ of ϵ values for which

$$\epsilon_n < \epsilon_0, \qquad \lim \epsilon_n = \epsilon_0.$$

On the other hand, according to Lemma 3 there exists a sequence $\{C_n\}$ of continuous rectifiable curves such that, if L_n denotes the length of C_n, we have

(a) $\lim L_n=L$,

(b) C_n is ϵ_n-ergodic to C_{ϵ_0} (and therefore ϵ_n-ergodic to M) with $\epsilon_n<\epsilon_0$ and $\lim \epsilon_n=\epsilon_0$.

Therefore, since $\Lambda(\epsilon_n)\leqq L_n$, it follows that

$$\lim \Lambda(\epsilon_n) \leqq L,$$

which is a contradiction to our previous result. Hence $\Lambda(\epsilon)$ is continuous on the left for $\epsilon=\epsilon_0$.

The case $\epsilon_0<\rho$ is now disposed of. There remains the case $\epsilon_0=\rho$ to consider. That $\Lambda(\epsilon)$ is continuous on the left in this case is trivial. The ergodic curve C shrinks to a point P, since $\Lambda(\rho)=0$, and one replaces the curve L_n above by a circle of radius $1/n$ described about P as center.

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