INVOLUTORIAL SPACE TRANSFORMATIONS ASSOCIATED WITH A RATIONAL RULED SURFACE*

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1. *Introduction.* The present paper concerns various new types of Cremona involutorial transformations in S_3. Each one is a sample of an infinite category, and the concept can be extended to higher spaces.

Transformations obtained by joining corresponding points of curves or by associating the points of a curve with a projective pencil of surfaces have been studied from time to time and are sketched in the *Encyklopädie*, the *Repertorium*, and particularly in *Selected Topics of Algebraic Geometry*, including the Supplementary Report (Bulletin 63, National Research Council, 1928, No. 96, 1934). All these papers can be found in *Topics*, Chapters 8, 9, or Supplement, Chapters 4, 5 under the names Black, Carroll, Davis, DePaoli, Dye, Moffa, Montesano, Sharpe, and Snyder.

The types here discussed can not be put into any of the categories previously mentioned. The complex of lines defined by pairs PP' of associated points is not linear, is not special, and does not have any particular role in the problem. In most of the earlier cases it was formed by the secants to a given curve, or was linear. In the DePaoli types the lines PP' describe a congruence, each line containing an infinite number of pairs of conjugate points.

The procedure used in this paper is to establish a (1, 1) correspondence between the generators of a ruled surface R and the surfaces of a pencil $|F|$. A general point P of space selects a surface F of the pencil and hence the associated generator r of R. The plane π determined by r and P is tangent to R at a point Q. The line PQ meets the surface F in P and a residual point P'; any other intersections are accounted for by properly relating $|F|$ and R. The points P, P' are an associated pair in an involutorial transformation under which the pencil $|F|$ and the congruence of conics cut from $|F|$ by the planes π are invariant.

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2. *Pencil of Quadrics through a* C_4, $(p=1)$. A pencil of quadric surfaces $|F_2|:C_4$ having an elliptic quartic curve in common, and a ruled surface $R_{n+1}:l^n$ of order $n+1$ with an n-fold line l determine an involutorial transformation of order $8n+5$. In this transformation there are two fundamental curves of the first species; one is the C_4 which is the base of $|F_2|$ and the other is a C_{2n+3} which is the locus of the piercing points of the lines r and their associated quadrics. The image of each of these points is a conic and the conics generate the image surface L_{8n+4} of C_{2n+3}.

There are $2n-2$ torsal generators on R_{n+1}. The tangent planes along them cut conics from the associated quadrics which are parasitic, since every point of such a generator may be taken as a point Q in the tangent plane, and hence the image of every point of the conic is the whole conic.

Through each generator r there are two planes which cut degenerate conics from the associated quadric. If the point of contact Q of such a plane is on one of the lines of the degenerate conic, then every point of the line will go into the whole line, and the line is parasitic. The number of such lines is $8n+8$ and is determined by taking the complete intersection of two homoloidal surfaces.

In any plane π the points of contact of the tangents drawn from Q to the conic section lie on the invariant surface K_{4n+4} of the transformation. When Q is a point of C_{2n+3}, then that point will lie on both K and L, and these surfaces will have contact along C_{2n+3}.

Let the equation of R_{n+1} be $ax_2+bx_3+cx_4=0$, where a, b, c are binary forms of order n in x_1, x_2. The equations of the line l are $x_1=x_2=0$, and any plane through l such as $\mu x_1-\lambda x_2=0$ cuts from R_{n+1} a line r whose equations may be written as follows: $\mu x_1-\lambda x_2=\bar{a}x_2+\bar{b}x_3+\bar{c}x_4=0$, where $\bar{a}=a(\lambda, \mu), \cdots$. The plane π determined by a point $P(y)$ and the line r is tangent to R_{n+1} at the point $Q(z)$ which has the coordinates $(\lambda A_0R, \mu A_0R, \mu B_0R-n\bar{c}M, \mu C_0R+n\bar{b}M)$, where $A_0\equiv\bar{b}_1\bar{c}_2-\bar{b}_2\bar{c}_1, \cdots$, and $\bar{a}_i\equiv\partial a/\partial x_i|_{\lambda,\mu}, \cdots$. The symbols R, M represent the forms $\mu y_1-\lambda y_2$, $\bar{a}y_2+\bar{b}y_3+\bar{c}y_4$, respectively.

Let the equation of the pencil of quadrics be $\mu F_2'-\lambda F_2''=0$. A point $P(y)$ determines a quadric F_2 which the line PQ meets in the residual point $P'(y')$, where $y_i'=y_iL-z_iK$, and $L\equiv F_2(z)$,

$K\equiv y_i\partial F(z)/\partial z_i$. The equations of the transformation are obtained by replacing λ, μ by $F_2'(y)$, $F_2''(y)$ in y_i'.

The table of characteristics of the transformation is

$$
\begin{aligned}
S_1 \sim S_{8n+5} &: C_4^{4n+1} + C_{2n+3}^{2+2t},\\
C_{2n+3} \sim L_{8n+4} &: C_4^{4n+1} + C_{2n+3}^{2+t},\\
C_4 \sim F_{16n+8} &: C_4^{8n+1} + C_{2n+3}^{4+4t},\\
K_{4n+4} &: C_4^{2n+1} + C_{2n+3}^{1+t}.
\end{aligned}
$$

The superscript t is used to indicate that the multiplicity is due to contact; for example, the tangent planes to the two sheets of an S_{8n+5} through C_{2n+3} coincide at each point of C_{2n+3} with the tangent plane to K_{4n+4} at the point. The nature of the surface F_{16n+8} is obtained by finding the transform of a quadric F_2.

3. *Pencil of Quadrics through the Line l.* If the basis curve of the pencil of quadrics consists of the line l and a residual space cubic, then the equation of $|F_2|$ may be written in the form $|F_2| = x_1(\alpha x)+x_2(\beta x)=0$, where (αx) is a linear homogeneous form in x_i and $\alpha_i\equiv\mu\alpha_i'-\lambda\alpha_i''$, $\cdots$. The form $F_2(z)$ is composite and has the factors $A_0R[\lambda(\alpha z)+\mu(\beta z)]\equiv A_0RL$. The equations of the transformation are $y_i'=y_iA_0RL-z_iK$, but $y_1'=A_0R^2[(\beta z)-\lambda A_0\{\lambda(\alpha y)+\mu(\beta y)\}]$, and $y_2'=A_0R^2[(\beta z)+\mu A_0\{\lambda(\alpha y)+\mu(\beta y)\}]$.

When a plane π passes through l, the point of contact $Q(z)$ is at the intersection of l and r. Such a plane cuts from F_2 a line meeting l elsewhere than at Q. The image of the line is Q, and as Q moves on l, the image lines generate the surface $R_3=0$ taken twice. At any point on l the tangent planes to the $2n$ sheets of a homoloidal surface coincide in pairs with the n tangent planes to $M_{2n+1}=0$ at that point, since the highest powers of y_3, y_4 have $(\bar{b}y_3+\bar{c}y_4)^2$ as a factor in the equation of an S_{8n+5}, and $(\bar{b}y_3+\bar{c}y_4)$ in the case of L_{4n+5}. The invariant surface K_{4n+4} does not have contact along l, but in the intersection of K_{4n+4}, and on S_{8n+5} the line l must be counted an additional n times.

In this transformation each generator r meets its associated quadric in a point on l, and one residual point. The locus of these residual points is a fundamental curve C_{n+3}, $(p=0)$, which has the surface L_{4n+5} as its image.

The $2n-2$ pinch points of R_{n+1} along l are fundamental points of the transformation, since each of these points is associated with all the planes through its torsal generator. The image of a pinch point is the quadric surface associated with the torsal generator through the point. The equations of these quadrics are obtained by putting A_0 equal to zero.

The number of parasitic lines due to the point $Q(z)$ lying on a degenerate conic cut from F_2 by a plane π is $4n+5$, and the number of parasitic conics is $2n-2$ as before. The table of characteristics of the transformation is

$$
\begin{aligned}
S_1 \sim S_{8n+5} &: l^{4n+2} + n\bar{l}^{2} + C_3^{4n+1} + C_{n+3}^{2+2t} + (2n-2)O_i^{4n+5},\\
l \sim 2R_3 &: l^{2} \qquad\qquad + C_3 \qquad + C_{n+3} + (2n-2)O_i^{2},\\
C_{n+3} \sim L_{4n+5} &: l^{2n+2} + n\bar{l} + C_3^{2n+2} + C_{n+3}^{1+t} + (2n-2)O_i^{2n+3},\\
C_3 \sim F_{12n+6} &: l^{6n+1} + n\bar{l}^{4} + C_3^{6n+2} + C_{n+3}^{2+2t} + (2n-2)O_i^{6n+5},\\
(2n-2)O_i \sim A_{4n-4} &: l^{2n-2} \qquad\qquad + C_3^{2n-2} \qquad\qquad + (2n-2)O_i^{2n-1},\\
K_{4n+4} &: l^{2n+2} \qquad\qquad + C_3^{2n+1} + C_3^{1+t} + (2n-2)O_i^{2n+3},
\end{aligned}
$$

where the notation $\bar{l}$ is used to indicate contact along l.

4. *Pencil of Cubic Surfaces with a Double Line.* A pencil of cubic surfaces $|F_3|$ having a double line l and a rational twisted quintic C_5 as a base is made projective with the pencil of planes $|p|$ through l. A general point P will determine a surface F_3 and the plane p which corresponds to F_3. The plane p will cut from F_3 a line r which generates a ruled quartic surface $R_4{:}l^3$. The pencil $|F_3|$ and the surface R_4 determine a transformation in the manner described in the introduction.

Let the equation of the pencil of cubic surfaces be $ax_2+bx_3+cx_4=0$, where a, b, c are binary quadratic forms in x_1, x_2, and $a_{ij}\equiv\mu a_{ij}' -\lambda a_{ij}'', \cdots$. The equations of the planes p and of the line r have the same form as those used in §2. The ruled quartic R_4 has the equation $x_2F_3' -x_1F_3'' =0$. The plane π determined by a point $P(y)$ and the line r is tangent to R_4 at the point $Q(z)$, where

$$(z_i)\equiv(\lambda AR,\ \mu AR,\ \mu BR-2\bar{c}M,\ \mu CR+2\bar{b}M),$$

in which $A\equiv A_0+2A^0, \cdots$, and $A^0\equiv\bar{b}'\bar{c}''-\bar{b}''\bar{c}', \cdots$.

The residual intersection of the line PQ with the associated surface F_3 is the point $P'(y')$, where

$$y_i' = y_i\left(z_i \frac{\partial F(y)}{\partial y_i}\right) z_i \left(y_i \frac{\partial F(z)}{\partial z_i}\right).$$

Replacing λ, μ by $F'(y)$, $F''(y)$, we have $y_i' = R^3(y_i AL - z_i K)$. The forms L, K can be expressed as follows:

$$\begin{aligned} L &\equiv \bar{a}_1 A^0 + \bar{b}_1 B^0 + \bar{c}_1 C^0 - A^0 N, \\ N &\equiv 2\mu y_2 \alpha - (\mu y_1 + \lambda y_2)\beta + 2\lambda y_1 \gamma, \\ K &\equiv N[2\lambda y_1 A_{22} - (\mu y_1 + \lambda y_2) A_{12} + 2\mu y_2 A_{11}] \\ &\quad + 2\mu RD - A(\alpha y_2^2 - \beta y_1 y_2 + \gamma y_1^2). \end{aligned}$$

The letters α, β, γ represent the second-order determinants of the matrix

$$\left\| \begin{matrix} a_{11}'y_2 + b_{11}'y_3 + c_{11}'y_4 & a_{12}'y_2 + b_{12}'y_3 + c_{12}'y_4 & a_{22}'y_2 + b_{22}'y_3 + c_{22}'y_4 \\ a_{11}''y_2 + b_{11}''y_3 + c_{11}''y_4 & a_{12}''y_2 + b_{12}''y_3 + c_{12}''y_4 & a_{22}''y_2 + b_{22}''y_3 + c_{22}''y_4 \end{matrix} \right\|,$$

and A_{ij} are the cofactors of a_{ij} in $D \equiv | a_{11} b_{12} c_{22} |$.

When a point Q is at the intersection of r and C_5, the plane π is tangent to both R_4 and F_3, since the line r and the tangent to C_5 determine the plane π. The image of each point of C_5, when it plays this role, is the residual conic cut from F_3 by π. These conics generate the surface L_{18}. The C_5 has another image surface, however, since it also enters the transformation as a part of the base of $|F_3|$. This other image surface is obtained by transforming an F_3 and is of order 48. Geometrically this surface is generated by the images of the four points of C_5 not on r in any plane π.

In this transformation each line r is a parasitic line and the surface R_4 generated by these lines factors out of the transformation three times. The line l is a multiple parasitic line along which S_{31} and L_{18} have contact, since the highest powers of y_3, y_4 in their equations have N_6 as a factor.

The surfaces L_{18} and K_{16} have contact along C_5 as in the other transformations. The four pinch points O_i of R_4 along l are fundamental, and the tangent planes along the four torsal generators cut conics from the associated F_3's which are parasitic in the transformation. The number of parasitic lines is 14 as determined by the complete intersection of two S_{31}'s. The table of characteristics of the transformation is

$$
\begin{aligned}
S_1 &\sim S_{31} : l^{19} + 3\bar{l} + C_5^{10+2t} + 4O_i^{22}, \\
C_5 &\sim L_{18} : l^{11} + 3\bar{l} + C_5^{6+t} + 4O_i^{12}, \\
&\quad F_{48} : l^{28} + 3\bar{l}^{2} + C_5^{15+5t} + 4O_i^{34}, \\
4O_i &\sim A_{12} : l^{8} \qquad + C_5^{4} + 4O_i^{9}, \\
&\quad K_{16} : l^{10} \qquad + C_5^{5+t} + 4O_i^{11}.
\end{aligned}
$$

5. *Pencil of Cubic Surfaces with Two Double Points.* The transformation to be developed here is similar to the one in the preceding section. A pencil of cubic surfaces $|F_3|$ through two double points O_i contains the line through them and has a residual basis curve C_8 of order 8 with triple points at O_i. Each F_3 is tangent to a fixed plane along the line l through O_i. These tangent planes cut lines r from their associated F_3 and the lines r generate a ruled quartic surface $R_4 : l^3$.

Let $ax_2+bx_3+cx_4+(\mu x_1-\lambda x_2)x_3x_4=0$ be the equation of the $|F_3|$. The equations of the lines r and the surface R_4 and the coordinates of the points $Q(z)$ are the same as those in the last section.

When λ, μ are replaced by $F_3'(y)$, $F_3''(y)$, the form M is factorable into R and another factor, which is of the sixth order in y and contains a term $(y_3y_4)^2$. The factor R^3 comes off from the equations of the transformation and we have $y_i' = y_iL - z_iK$.

The curve C_8 meets each generator r in two points at which R_4 and the associated F_3 have contact. The image of C_8 consists of two surfaces; one is the L_{30} due to the contact of $|F_3|$ with R_4 along C_8, and the other surface is an F_{60} due to the C_8 being a part of the base of $|F_3|$.

The lines r are all parasitic and the line l is a multiple parasitic line. There are 24 parasitic lines which are not generators of R_4, and 4 parasitic conics. The surfaces L_{30} and S_{31} have contact along l, and the surfaces K_{16} and L_{30} have contact along C_8. The table of characteristics of the transformation is

$$
\begin{aligned}
S_1 &\sim S_{31} : l^{6} + 6\bar{l} + C_8^{10+2t}, \\
C_8 &\sim L_{30} : l^{6} + 6\bar{l} + C_8^{10+t}, \\
&\quad F_{60} : l^{11} + 6\bar{l}^{2} + C_8^{19+5t}, \\
&\quad K_{16} : l^{4} \qquad + C_8^{5+t}.
\end{aligned}
$$

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