ON POINCARÉ'S RECURRENCE THEOREM

BY CORNELIS VISSER

1. *Introduction.* Let S be a space in which is defined a measure μ such that $\mu(S)=1$. Suppose we are given a one parameter group of one to one transformations T_t, $(-\infty<t<\infty)$, of S into itself, with the properties:

$$T_sT_t = T_{t+s}. \tag{1}$$

(2) For any measurable set E and any t the set T_tE is measurable and $\mu(T_tE)=\mu(E)$.

The following extension of Poincaré's recurrence theorem was proved by Khintchine.*

For any measurable E and any $\lambda<1$,

$$\mu(E\cdot T_tE) \geqq \lambda(\mu(E))^2$$

for a set of values t that is relatively dense on the t axis.

In this paper we give an elementary proof of this statement.

2. *An Auxiliary Theorem.* We prove the following theorem from which the recurrence theorem is an immediate consequence and which is also interesting in itself.

Let S be a space with a measure μ such that $\mu(S)=1$ and let $E_1, E_2, \cdots$ be an infinite sequence of measurable sets in S, all having a measure not less than m. Then for any $\lambda<1$ there exist in the sequence two sets E_i and E_k such that

$$\mu(E_iE_k) \geqq \lambda m^2.$$

Let us suppose that $\mu(E_iE_k)<p$ for any i and k. If we put

$$F_1 = E_1, \quad F_2 = E_2 - E_2F_1, \quad F_3 = E_3 - E_3F_2 - E_3F_1,$$
$$\cdots, \quad F_n = E_n - E_nF_{n-1} - \cdots - E_nF_1,$$

no two of the sets F have common points and F_i is part of E_i. Therefore

$$\mu(F_1) \geqq m, \quad \mu(F_2) > m - p, \quad \mu(F_3) > m - 2p,$$
$$\cdots, \quad \mu(F_n) > m - (n-1)p,$$

* A. Khintchine, *Eine Verschärfung des Poincaréschen "Wiederkehrsatzes,"* Compositio Mathematica, vol. 1 (1934), pp. 177–179.

and thus for $n=1, 2, \cdots$,

$$\mu(F_1 + \cdots + F_n) = \mu F_1 + \cdots + \mu F_n \geqq nm - \frac{1}{2} n(n-1)p.$$

It follows that

$$1 = \mu(S) \geqq \mu(F_1 + \cdots + F_n) \geqq nm - \frac{1}{2} n(n-1)p.$$

We now choose n such that

$$\frac{m}{p} < n \leqq \frac{m}{p} + 1.$$

Then we obtain

$$\begin{aligned} 1 &\geqq \frac{m}{p} m - \frac{1}{2}\left(\frac{m}{p} + 1\right)\frac{m}{p} p \\ &= \frac{m^2}{2p} - \frac{m}{2}, \end{aligned}$$

or

$$p \geqq \frac{m^2}{2+m}.$$

Hence, if we exclude the trivial case $m=1$,

$$p > \frac{1}{3} m^2.$$

From this it follows that there must be two sets E_i and E_k with

$$\mu(E_i E_k) \geqq \frac{1}{3} m^2.$$

We shall now prove that the factor $1/3$ may be replaced by an arbitrary $\lambda < 1$. We consider the product space S^n, formed by the systems $(x_1, \cdots, x_n)$ of n points in S, and in this product space the sequence of sets $E_1^n, E_2^n, \cdots$. In S^n we can define a measure $\bar{\mu}$ such that the product of n measurable sets of S is measurable and has a measure that equals the product of the measures of its components. In applying the result we just obtained to S^n and the sequence $E_1^n, E_2^n, \cdots$, we find two sets E_i^n and E_k^n such that

$$\bar{\mu}(E_i^n E_k^n) \geqq \frac{1}{3}(m^n)^2.$$

Now

$$\bar{\mu}(E_i^n E_k^n) = \bar{\mu}((E_i E_k)^n) = (\mu(E_i E_k))^n,$$

and consequently

$$(\mu(E_i E_k))^n \geqq \frac{1}{3}(m^n)^2,$$

or

$$\mu(E_i E_k) \geqq \left(\frac{1}{3}\right)^{1/n} m^2.$$

Given $\lambda < 1$, we can always define n such that $(1/3)^{1/n} \geqq \lambda$ and then select the pair E_i, E_k. This proves the theorem.

3. *Proof of the Recurrence Theorem.* Assume the contrary: There is a measurable set E and a number $\lambda < 1$ such that

$$(*) \qquad \mu(E \cdot T_t E) < \lambda(\mu(E))^2$$

on arbitrarily large t-intervals. Let I_1 be a closed interval on which (*) holds; denote by $2l_1$ its length and by l_2 its center. There is an interval I_2 on which (*) holds and which has a length $> 2(l_1 + |l_2|)$. Denote by l_3 the center of I_2 and by I_3 an interval on which (*) holds and which has a length $> 2(l_1 + |l_2| + |l_3|)$, and so forth. Then the numbers $l_k - l_i$, $(i < k)$, belong to the intervals I_{k-i}; hence for any i and k, $(i < k)$,

$$\mu(E \cdot T_{l_k - l_i} E) < \lambda(\mu(E))^2,$$

and consequently

$$\mu(T_{l_i} E \cdot T_{l_k} E) < \lambda(\mu(E))^2$$

in contradiction to the theorem of §2, applied to the sequence $T_{l_1}E, T_{l_2}E, \cdots$. This proves the recurrence theorem.

It will be seen that it is not essential that t in T_t is a continuous parameter. The same method gives the same result in the case that t only runs through the values $0, \pm 1, \pm 2, \cdots$.

4. *Remark.* Let the sequence $E_1, E_2, \cdots$ be as in §2. Then we can even assert *that for every* $\lambda<1$ *there exists an infinite subsequence* $E_{i_1}, E_{i_2}, \cdots$ *such that for every* p *and* q

$$\mu(E_{i_p}E_{i_q}) \geqq \lambda m^2.$$

We show first that there exists an infinite subsequence $E_{k_1}, E_{k_2}, \cdots$ such that $\mu(E_{k_1}, E_{k_p}) \geqq \lambda m^2$ for every p. Suppose that no such subsequence exists; then to every $n=1, 2, \cdots$ belongs a p_n such that

$$\mu(E_nE_m) < \lambda m^2 \quad \text{for} \quad m \geqq n + p_n.$$

Writing $n_1=1$, $n_2=n_1+p_{n_1}$, $n_3=n_2+p_{n_2}$, $\cdots$, we have then for every i and k,

$$\mu(E_{n_i}E_{n_k}) < \lambda m^2,$$

which contradicts the theorem of §2. The proof is now easily completed by applying the diagonal principle.

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ON THE ZEROS OF THE DERIVATIVE OF A RATIONAL FUNCTION*

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1. *Introduction.* The primary object of this note is to give a simple solution of a problem already discussed by many authors including the present one.† It is the problem of determining the regions within which lie the zeros of the derivative of a rational function when the zeros and poles of the function lie in prescribed circular regions.

THEOREM 1.‡ *For* $j=0, 1, \cdots, p$ *let* r_j *and* σ_j *be real constants*

* Presented to the Society, September 4, 1934.

† For an expository account and list of references see M. Marden, American Mathematical Monthly, vol. 42 (1935), pp. 277–286, hereafter referred to as Marden I.

‡ See M. Marden, Transactions of this Society, vol. 32 (1930), pp. 81–109, hereafter referred to as Marden II.