LINEAR CONNECTIONS OF NORMAL SPACE TO A VARIETY IN EUCLIDEAN SPACE*

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1. *Introduction.* This paper deals with an extension of the Gauss formulas of a surface imbedded in ordinary space to apply to an m-dimensional variety imbedded in an n-dimensional euclidean space. The Gauss and Codazzi relations are extended to give integrability conditions in terms of the Christoffel symbols of the second kind of the variety, a set of tensors corresponding to the coefficients of the second fundamental form of a surface in three space and a set of non-covariant quantities: the connection coefficients of the normal space.

Several authors† have considered generalizations of the Frenet or the Gauss formulas to apply to varieties lying both in euclidean and in more general spaces, but the nature of some of the coefficients which may appear in these formulas seems to have escaped serious study.

In the considerations here there will be encountered an imbedding euclidean space of n dimensions, a variety of m dimensions lying in it and at each point of this variety an m-dimensional tangent space and an $(n-m)$-dimensional normal space. A vector in the imbedding space will be denoted simply by a letter, and all indices running from 1 to n will be suppressed. Latin indices lying between a and k will have the range from 1 to m and Latin letters from p through z when used as indices will have the range from 1 to $(n-m)$. Corresponding Greek letters will be used as summation indices.

The work in this paper differs from that in most of the previ-

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† See, for instance, Voss, Mathematische Annalen, vol. 16 (1880), p. 129; H. Weyl, Mathematische Zeitschrift, vol. 12 (1922), p. 162; Schouten and van Kampen, Mathematische Annalen, vol. 105 (1931), p. 144; E. Bortolotti, Rendiconti del R. Istituto Lombardo di Scienze e Lettere, (2), vol. 64 (1931), p. 441; E. H. Cutler, Transactions of this Society, vol. 33 (1931), p. 832; C. E. Weatherburn, Reports of the Australian and New Zealand Association for the Advancement of Science, vol. 21 (1933), p. 12; Duschek and Mayer, *Lehrbuch der Differentialgeometrie*, vol. 2.

ously published papers in the method of selecting coordinates in the normal space; in this paper a set of perfectly arbitrary mutually orthogonal unitary vectors is set up as a coordinate system in the normal space at every point of the variety, while in most of the other papers the normal space is divided into several subspaces (the first containing all components of the first derivatives of the tangent vectors lying in the normal space, the second determined by additional components of the second derivatives, and so on) and in each of these subspaces an arbitrary coordinate system is introduced. Weyl derived the same Gauss formulas and integrability conditions we shall use, but he seems not to have noticed the properties of the coefficients which we shall call N^q_{pk}.

2. *The Gauss Formulas and Integrability Conditions.* A variety of m dimensions may be represented by a set of equations depending on m parameters, $x = x(u^k)$. In general there is a uniquely determined flat m-space tangent to this variety at every point; this is the linear space depending on the vectors $p_i \equiv \partial x/\partial u^i$; it is m-dimensional and uniquely determined if the vectors p_i are linearly independent. Also at every point there is a flat space which is absolutely perpendicular to the tangent space. If the tangent space is actually m-dimensional, the normal space is $(n-m)$-dimensional and contains sets of $(n-m)$ unit mutually orthogonal vectors. We may choose any such set and name it t_p. Now, any vector connected with a point of the variety may be written as a linear combination of the vectors p_i and t_p associated with that point. In particular we can write

$$(1) \qquad \frac{\partial p_i}{\partial u^k} = \Gamma^{\alpha}_{ik} p_\alpha + L^{\rho}_{ik} t_\rho,$$

where the coefficients Γ^k_{ij} are the Christoffel symbols of the second kind connected with the variety and the choice of parameters.

We write also the equations for the partial derivatives of the vectors t_p,

$$(2) \qquad \frac{\partial t_p}{\partial u^k} = - L_k^{\alpha p} p_\alpha + N^{\rho}_{pk} t_\rho,$$

where it is easily shown that $L^p_{ik} = g_{i\alpha} L_k^{\alpha p}$. Because the vectors t_p are mutually orthogonal unitary vectors, the coefficients N^q_{pk} must be anti-symmetric in the indices p and q.

The simplest method of arriving at integrability conditions seems to be that of variation of product integrals as developed by Rainich and Vaughan.* It is well known that we can take any set of equations of the form of (1) and (2) and by applying product integrals over some path arrive at a set of values for the vectors p_i and t_p along the path. However, for the variety to be a true variety it is necessary and sufficient that the vectors obtained for any point be the same no matter what path is used in deriving them; that is, the variation of the product integral used in their derivation must be zero. If we denote the matrices of the coefficients of the p_i and t_p in the expression for the partial derivatives with respect to u^k in (1) and (2) by A_k, a necessary and sufficient condition for the variation to vanish is

$$A_i A_k - A_k A_i + \frac{\partial A_i}{\partial u^k} - \frac{\partial A_k}{\partial u^i} = 0.$$

If we denote the matrices which make up A_k in the obvious manner by Γ_k, L_k, $L_k^{\cdot}$, and N_k, three relations analogous to the Gauss and Codazzi relations follow, conditions necessary and sufficient for the set of equations (1) and (2) to represent a true variety:

$$R^{\cdot}_{\cdot ki} \equiv \frac{\partial \Gamma_i}{\partial u^k} - \frac{\partial \Gamma_k}{\partial u^i} + \Gamma_i \Gamma_k - \Gamma_k \Gamma_i = L_k L_i^{\cdot} - L_i L_k^{\cdot},$$

$$S^{\cdot}_{\cdot ki} \equiv \frac{\partial N_i}{\partial u^k} - \frac{\partial N_k}{\partial u^i} + N_i N_k - N_k N_i = L_k^{\cdot} L_i - L_i^{\cdot} L_k, \tag{3}$$

$$\frac{\partial L_i}{\partial u^k} - \frac{\partial L_k}{\partial u^i} + \Gamma_i L_k - \Gamma_k L_i + L_i N_k - L_k N_i = 0.$$

Having given the quantities Γ_k, L_k, and N_k satisfying these conditions and having given also a set of values for the components of the fundamental metric tensor at some point on the variety, the variety may be completely reconstructed except for position by means of a product integration followed by a Riemann integration.

* Abstract, this Bulletin, vol. 40 (1934), p. 233.

3. *Laws of Transformation.* All the coefficients of the equations (1) and (2) are invariant under transformations of the coordinate system of the imbedding euclidean space. Under transformations of the system of parameters u^k, the coefficients Γ_{ij}^k transform under the well known law for the transformation of connection coefficients; under these transformations, the coefficients L_{ik}^p transform as second rank doubly covariant tensors with indices i and k, while the similar quantities with one index raised must obviously transform as second rank mixed tensors. These transformations are all obvious from the form of equation (1). From equation (2) it is also evident that under arbitrary changes in parameterization of the variety the coefficients N_{pk}^q transform as first rank covariant tensors with index k.

In the normal space we allow rotations; more general transformations could be introduced with slight changes in the formulas already developed, but the introduction of such additional generality would simply necessitate the distinction between covariance and contravariance without rendering any useful service. We consider transformations of the type $\bar{t}=At$, where A is an orthogonal matrix. Applying this to the quantities of equation (1), we get the law of transformation for the coefficients L_{ik}^p,

$$\bar{L}_{ik} = AL_{ik}.$$

By continuing to equations (2), we find the law of transformation of the other coefficients,

$$\bar{N}_k = AN_kA^T - A\frac{\partial A^T}{\partial u^k}.$$

This law of transformation is formally exactly the same as the law which must be obeyed by coefficients of linear connection. This permits the statement of the following theorem concerning differentiation of complete tensors.*

THEOREM. *The operation of covariant differentiation on a complete tensor, in which operation the differentiation is with respect to one of the parameters u^k, and in which the quantities N_{pk}^q are used*

* That is, a set of quantities which are components of tensors in all the coordinate systems we are considering; the process we describe has been called complete differentiation (see E. H. Cutler, loc. cit.).

as connection coefficients for indices applying to the normal space, and in which the ordinary Christoffel symbols, Γ^k_{ij}, are used as connection coefficients for indices appertaining to the tangent space, yields a complete tensor.

The expression S^p_{qki} of the second of the integrability conditions (3) is simply the Riemann tensor of the connection coefficients of the normal space. The analogy between the first and the second of these conditions is striking. The third of these integrability conditions is rewritten under the definition of covariant differentiation just stated as

$$\Delta_k L^p_{ij} - \Delta_i L^p_{kj} = 0.$$

The anti-symmetry of the connection coefficients of the normal space reiterates the equivalence of covariance and contravariance.

4. *Vanishing of the Riemann Tensors.* It may be that there exists a choice of the vectors t_p such that the coefficients N^q_{pk} vanish simultaneously; from the law of transformation of these coefficients, it is evident that a necessary and sufficient condition for the existence of such coordinates is that the coefficients have the form $N_k = -A(\partial A^T/\partial u^k)$; or, since the matrix A is orthogonal, $\partial A/\partial u^k = N_k A$. This equation is of the type whose solution for the matrix A may be expressed by product integrals; we can find a solution over any particular path, but the solution must not depend on the path; that is, the variation of the product integral must vanish. A necessary and sufficient condition for this is the vanishing of the Riemann tensor of the normal connection coefficients, $S^p_{qki}=0$. The solution we obtain by means of such an integration for A is an orthogonal matrix, for it can be shown that the product integral of any anti-symmetric matrix is orthogonal if the matrix used as an arbitrary constant of integration is orthogonal. Hence, *the vanishing of the Riemann tensor S^p_{qki} is a necessary and sufficient condition for the existence of coordinate systems for which all the N^q_{pk} vanish.*

The implications of the vanishing of this tensor are readily seen from the second of the integrability conditions (3); if we choose the parameters u^i in such a way that at a point in which we are interested the p_i are unit mutually orthogonal vectors, then $L_k^{ip} = L_{ki}^p$, and the integrability condition becomes

$$\begin{vmatrix} L_{i\alpha}^{r} & L_{k\beta}^{r} \\ L_{i\beta}^{s} & L_{k\alpha}^{s} \end{vmatrix} = 0.$$

Now rotate the vectors so that one of the matrices, say $L_{..}^{r}$, becomes a diagonal matrix, and the equation becomes

$$L_{ik}^{s}(L_{ii}^{r} - L_{kk}^{r}) = 0.$$

This means that *the vanishing of the tensor S_{qki}^{p} implies that at every point on the variety there exists a set of mutually orthogonal vectors in the tangent space of such a nature that all the principal directions of all the Dupin indicating quadrics lie along these vectors.* This is evident when we notice that the equation implies either that the other matrices $L_{..}^{s}$ are all diagonal when $L_{..}^{r}$ is or that $L_{ii}^{r} = L_{kk}^{r}$. If this last condition is true, the matrix $L_{..}^{r}$ has no principal directions with respect to the indices i and k, and we may rotate the vectors p_i and p_k in their plane until the matrix $L_{..}^{s}$ becomes a diagonal matrix without destroying the diagonal property of $L_{..}^{r}$.

In the case of a surface in a euclidean four-space we can get more definite results. The vanishing of the tensor S_{qki}^{p} implies that the indicatrix of Wilson and Moore* degenerates into a straight line. Indeed, if we choose the coordinate axis y along the vector t_1 and the axis z along t_2, the indicating conic may be written parametrically

$$y = \overset{1}{L}_{11} \cos^2\theta + 2\overset{1}{L}_{12} \sin\theta\cos\theta + \overset{1}{L}_{22} \sin^2\theta,$$

$$z = \overset{2}{L}_{11} \cos^2\theta + 2\overset{2}{L}_{12} \sin\theta\cos\theta + \overset{2}{L}_{22} \sin^2\theta.$$

Because of integrability conditions and the vanishing of the tensor S_{qki}^{p}, these reduce immediately to the equation of the line

$$\begin{vmatrix} x - \overset{1}{L}_{11} & y - \overset{2}{L}_{11} \\ \overset{1}{L}_{12} & \overset{2}{L}_{12} \end{vmatrix} = 0.$$

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* Wilson and Moore, Proceedings of the American Academy of Arts and Sciences, vol. 52 (1916), p. 324.