is solvable if and only if

$$\rho_m(M) \equiv \sum_{i=0}^{k-m} M^{p^{ni}} \frac{L_{m+i-1}}{L_i} \equiv 0.$$

It will be remarked that in general the first criterion is more useful. However, the latter is of some interest in itself. Furthermore, it suggests possible criteria for more general classes of congruences; I hope to develop the matter elsewhere.

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A DIFFERENTIAL EQUATION FOR APPELL POLYNOMIALS†

BY I. M. SHEFFER

By a *set* of polynomials $\{P_n(x)\}$, $(n = 0, 1, 2, \cdots)$, we shall mean an infinite sequence in which $P_n(x)$ is of degree *exactly*‡ n. Corresponding to a given set $\{P_n\}$ there are infinitely many sequences of polynomials $\{L_n(x)\}$ (with $L_n(x)$ of degree not exceeding n) and sequences of numbers $\{\lambda_n\}$ such that $\{P_n\}$ satisfies the linear differential equation (usually of infinite order) with parameter:§

$$(1) \qquad L[y(x)] \equiv \sum_{n=0}^{\infty} L_n(x) y^{(n)}(x) = \lambda y(x),$$

which for $\lambda = \lambda_n$ gives $P_n(x)$. In fact, suppose $\{P_n\}$ is given. Let $\{\lambda_n\}$ be any sequence of numbers subject only to the condition that λ_n is not identically zero in n. Then a unique sequence $\{L_n(x)\}$ exists such that $L[P_n(x)] = \lambda_n P_n(x)$, $(n = 0, 1, \cdots)$, where not all the L_n's are identically zero, and where no $L_n(x)$ is of degree exceeding n. The polynomial $L_n(x)$ is readily obtained by recurrence from $L_0, \cdots, L_{n-1}$. If we write

$$(2) \qquad L_n(x) = l_{n0} + l_{n1}x + \cdots + l_{nn}x^n,$$

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† Presented to the Society, April 25, 1935.

‡ For many purposes it suffices to have $P_n(x)$ of degree not exceeding n. Here, however, it is convenient to use the stricter condition.

§ See Sheffer, American Journal of Mathematics, vol. 53 (1931), pp. 29–30, for a relation suggestive of (1).

where l_{nn} and other coefficients may be zero, we shall have

(3) $\qquad \lambda_n = l_{00} + nl_{11} + n(n - 1)l_{22} + \cdots + n!l_{nn}.$

If we *start* with equation (1), that is, with the coefficients $\{L_n(x)\}$, and ask for what values of λ there exists a polynomial solution, we find as a necessary condition that λ have one of the values (3). Conversely, if λ has any value λ_n of (3), then there will correspond a polynomial solution $P_n(x)$ which is unique to within a multiplicative constant, provided $\lambda_m \neq \lambda_n$, $(m \neq n)$. Moreover, $P_n(x)$ will be of degree exactly n, so that $\{P_n\}$ is a *set*. In the case in which $\lambda_m = \lambda_n$ for two (or more) distinct values of m and n, there may or may not be, for this common characteristic number, as many linearly independent polynomial solutions as the order of the characteristic number.

DEFINITION. Let us suppose that equation (1) is given. If for each λ_n given by (3) there is a polynomial solution $P_n(x)$ of degree (exactly) n, we shall say that equation (1) is *non-singular*.

COROLLARY. *Equation* (1) *is non-singular if* $\lambda_m \neq \lambda_n$, $(m \neq n)$.

If equation (1) is non-singular, but $\lambda_m = \lambda_n$ for some $m \neq n$, then if P_m and P_n are corresponding solutions of degree m and n, the polynomial $aP_m + bP_n$ is also a solution, for every choice of a and b. In this case the equation does not define a *unique*† set of polynomials $\{P_n\}$.

Because of the property of equation (1) that *any* set $\{P_n(x)\}$ is a set of solutions, (for a suitably chosen sequence $\{L_n(x)\}$), it seems appropriate to refer to (1) as a *universal type equation for sets of polynomials*. This comprehensive character of a type equation suggests that it may prove useful in the study of properties of sets of polynomials. We may propose the following program. Given a set of polynomials, or a class of sets; to characterize the type equations that they satisfy, and, in particular, to determine if (and when) they correspond to a type equation of *finite order*.

The present note investigates these questions for the class of Appell sets of polynomials.

The set $\{P_n\}$ is an Appell set if we have the relations

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† We ignore the obvious arbitrary multiplicative constant attached to each $P_n(x)$.

(4) $$\frac{dP_n(x)}{dx} = P_{n-1}(x), \qquad (n = 1, 2, \cdots).$$

The following are well known characterizations of Appell sets.

(i) *A necessary and sufficient condition that $\{P_n\}$ be an Appell set is that a formal power series*

(5) $$A(t) \sim \sum_0^\infty a_n t^n$$

exist such that

(6) $$e^{tx} A(t) \sim \sum_0^\infty P_n(x) t^n.$$

We shall refer to $A(t)$ as the *generating function* (or power series) for $\{P_n(x)\}$.

(ii) *Set $\{P_n\}$ is an Appell set if and only if there exists a sequence of numbers $\{a_n\}$ such that*†

(7) $$P_n(x) = a_0 \frac{x^n}{n!} + a_1 \frac{x^{n-1}}{(n-1)!} + \cdots + a_n \frac{x^0}{0!},$$

$$(n = 0, 1, \cdots).$$

THEOREM 1. *Let $\{P_n\}$ be any Appell set, with generating function $A(t)$. Define the (formal) power series*

(8) $$B(t) \sim \sum_0^\infty b_n t^n$$

by

(9) $$B(t) = \frac{A'(t)}{A(t)}.$$

Then $\{P_n\}$ satisfies the equation of type (1):

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† In both (6) and (7), in order that $\{P_n\}$ be a true set, that is, for P_n to be of degree exactly n, it is necessary (and sufficient) that $a_0 \neq 0$. There is no great difficulty, however, if $a_0 = \cdots = a_k = 0$, $a_{k+1} \neq 0$. For then $P_n(x)$, as given by (6) and (7), is identically zero for $0 \leq n \leq k$, after which we get $P_n(x)$ of degree $n - (k+1)$; and if we define $B(t)$ by $A(t) = t^{k+1} B(t)$, then $b_0 \neq 0$, and the Appell set for $B(t)$ is the same set as for $A(t)$ after the first $(k+1)$ P's (which are identically zero) for $A(t)$ have been omitted.

$$L[P_n(x)] \equiv (b_0 + x)P_n'(x) + b_1 P_n''(x) + b_2 P_n'''(x) + \cdots$$

(10)

$$= nP_n(x), \qquad\qquad (n = 0, 1, \cdots).$$

Conversely, if the set $\{P_n\}$ satisfies the equation (10), *where $b_0, b_1, \cdots$ are constants, then (to within multiplicative constants) $\{P_n\}$ is an Appell set, and its generating function $A(t)$ is defined by* (9).

From (3) we see that $\lambda_n = n$ for equation (10). Therefore $\lambda_m \neq \lambda_n$, $(m \neq n)$, so that to each λ_n corresponds an essentially unique polynomial, and it is of degree n. Suppose the set $\{P_n\}$ satisfies (10). Let $P_0(x) = a_0$, and choose the multiplicative constants so that in $P_n(x)$ the coefficient of x^n is $a_0/n!$. It is verified that $P_1'(x) = P_0(x)$. Assuming that $P_k'(x) = P_{k-1}(x)$ for $k = 1, 2, \cdots, n-1$, we shall prove it for $k = n$. Differentiating (10), we find that $L[P_n'(x)] = (n-1)P_n'(x)$. Hence, since L is non-singular, we have $P_n'(x) = c_n P_{n-1}(x)$. Comparing coefficients of x^{n-1}, we find that $c_n = 1$. Hence $\{P_n\}$ is an Appell set. Let its generating function be $A(t) \sim \sum_0^\infty a_n t^n$, where $a_0 = P_0$ as already stated. Then $P_n(x)$ has the representation (7). If we substitute this in (10) and equate coefficients of like powers of x, we get the equations

$$b_0 a_{n-j-1} + b_1 a_{n-j-2} + \cdots + b_{n-j-1}\, a_0 = (n - j)a_{n-j},$$

$$(j = 0, 1, \cdots, n).$$

But these relations are completely equivalent to relation (9), as is easily verified. The converse part of Theorem 1 is thus established.

Now let $\{P_n\}$ be any Appell set, and let $A(t)$ be its generating function. Let us define $B(t)$ by (9), so that the coefficients of (10) are defined. Let $\{Q_n\}$ be the Appell set defined by (10), of whose existence (and uniqueness if we choose $Q_0(x) = \alpha_0$) we have just given proof. Then the generating function $A^*(t)$ for $\{Q_n\}$ is defined, by the part already proved, by the equation $B(t) = A^{*\prime}(t)/A^*(t)$. Since $A(0) = A^*(0) = a_0$, therefore $A^*(t) = A(t)$. Hence $Q_n(x) \equiv P_n(x)$, $(n = 0, 1, \cdots)$, so that $\{P_n\}$ satisfies (10). This completes the proof.

COROLLARY. *A necessary and sufficient condition that a set $\{P_n\}$ be an Appell set is that it satisfy the recurrence relations*

(11) $\quad nP_n(x) = (b_0 + x)P_{n-1}(x) + b_1P_{n-2}(x) + \cdots + b_{n-1}P_0(x),$

$$(n = 0, 1, \cdots).$$

For if $\{P_n\}$ is an Appell set, it satisfies (10) and therefore (11). Now suppose $\{P_n\}$ is a set satisfying (11). We verify at once that $P_1' = P_0$. Suppose $P_k' = P_{k-1}$, $(k = 1, 2, \cdots, n-1)$; to prove that $P_n' = P_{n-1}$. If we differentiate (11) and use our induction assumption, we get $nP_n' = nP_{n-1}$, so that $P_n' = P_{n-1}$, and the set $\{P_n\}$ is an Appell set.

Relations (10) and (11) give us two further characterizations of Appell sets. But they are not the only ones possible. For example, if we want an equation (for Appell sets) of type (1) with $\lambda_n = n(n-1)$, then as was proved in Theorem 1, we obtain the equation

(12) $\qquad L_{[2]}[P_n(x)] = n(n-1)P_n(x),$

where the operator $L_{[2]}$ is defined by the equation

$$L_2[y] \equiv \sum L_{[2]n}(x)y^{(n)}(x)$$

with

(13) $\quad L_{[2]}(x; t) \sim \sum_{n=0}^{\infty} L_{[2]n}(x)t^n = \frac{t^2}{A}[A'' + 2xA' + x^2A],$

where $A(t)$ is the generating function for $\{P_n\}$.

More generally, if $\{P_n\}$ is an Appell set defined by $A(t)$, then

(14) $\qquad L_{[k]}[P_n(x)] = n(n-1) \cdots (n-k+1)P_n(x),$

where

(15)
$$L_{[k]}(x; t) \sim \sum_{n=0}^{\infty} L_{[k]n}(x)t^n = \frac{t^k}{A}\left[A^{(k)} + \frac{k}{1!}xA^{(k-1)} \right.$$
$$\left. + \frac{k(k-1)}{2!}x^2A^{(k-2)} + \cdots + \frac{k!}{k!}x^kA\right].$$

Now suppose that we wish λ_n to be a polynomial in n. We can suppose that† $\lambda_0 = 0$. Then we have

(16)
$$\lambda_n = nl_{11} + n(n-1)l_{22} + \cdots$$
$$+ n(n-1) \cdots (n-k+1)l_{kk},$$

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† For this merely serves to alter the (constant) value of $L_0(x)$, and does not affect the polynomials $L_n(x)$ of index $n>0$.

where $l_{11}, \cdots, l_{kk}$ are constants (that is, independent of n). Exactly as in Theorem 1, we obtain the following result:

THEOREM 2. *Let* $\{P_n\}$ *be any Appell set and* $A(t)$ *its generating function. Let* $\{\lambda_n\}$ *be given by* (16) *for any* $k \geq 1$. *Then* $\{P_n\}$ *satisfies the equation*

$$(17) \qquad L[P_n(x)] \equiv \sum_{r=0}^{\infty} L_r(x) P_n^{(r)}(x) = \lambda_n P_n(x),$$

where L *is defined by*

$$(18) \qquad \begin{aligned} L(x; t) &\sim \sum_{r=0}^{\infty} L_r(x) t^r \\ &= l_{11} L_{[1]}(x; t) + l_{22} L_{[2]}(x; t) + \cdots + l_{kk} L_{[k]}(x; t), \end{aligned}$$

the functions $L_{[p]}(x; t)$ *being given by* (15). *Conversely, given* (17) *with* L *defined by* (18) *for any function* $A(t)$, *then there exists an Appell set of polynomials*† $\{P_n\}$ *satisfying* (17), *and its generating function is precisely* $A(t)$.

We proceed to characterize the equations of type (1) that define one and the same Appell set.

THEOREM 3. *Let* $\{P_n\}$ *be an Appell set, and* $A(t)$ *its generating function. A necessary and sufficient condition that the set* $\{P_n\}$ *be a solution of an equation of type* (1) (*with* $\lambda_0 = 0$, *which is no restriction*) *is that the operator* $L[y] \equiv \sum_{n=0}^{\infty} L_n(x) y^{(n)}(x)$ *be defined*‡ *by*

$$(19) \qquad L(x; t) \sim \sum_{n=0}^{\infty} L_n(x) t^n \sim \sum_{k=1}^{\infty} l_{kk} L_{[k]}(x; t),$$

where $L_{[k]}(x; t)$ *is given by* (15), *and the* l_{kk}'s *are constants. The characteristic numbers* $\{\lambda_n\}$ *are given by* (3).

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† If $\lambda_m \neq \lambda_n$, $(m \neq n)$, then this Appell set gives *all* the polynomial solutions of (17) (to within multiplicative constants). But if m and n (distinct) exist such that $\lambda_m = \lambda_n$, this Appell set does not include all the polynomial solutions of (17).

‡ Although the last series in (19) may not converge, yet it serves to determine a unique sequence of polynomials $L_1(x)$, $L_2(x)$, $\cdots$ which are the coefficients of equation (1). For $L_{[k]}(x; t)$, expressed as a power series in t, begins with a term in t^k. Hence $L_n(x)$, which is the coefficient of t^n in $L(x; t)$, is given by a *finite sum*, namely, the coefficient of t^n in the sum $\sum_{k=1}^{n} l_{kk} L_{[k]}(x; t)$.

First we shall show that for any sequence of numbers $\{l_{kk}\}$ the Appell set $\{P_n\}$ satisfies equation (1) with L defined by (19). Let

$$L^{[n]}(x; t) = \sum_{k=0}^{n} l_{kk} L_{[k]}(x; t) \sim \sum_{r=0}^{\infty} L_r^{[n]}(x) t^r,$$

where n is any positive integer. The remark of the last footnote shows that $L_r^{[n]}(x) = L_r(x)$, $(r = 0, 1, \cdots, n)$. Now, by Theorem 2, we have

$$L^{[n]}[P_r(x)] = \{r l_{11} + r(r-1) l_{22} + \cdots + r! l_{rr}\} P_r(x)$$

$$= \lambda_r P_r(x), \qquad\qquad (r = 0, 1, \cdots).$$

Hence

$$L[P_r(x)] = L^{[n]}[P_r(x)] = \lambda_r P_r(x), \qquad (r = 0, 1, \cdots, n);$$

and inasmuch as n can be chosen arbitrarily, it follows that $L[P_r(x)] = \lambda_r P_r(x)$ for *all* r. That is, $\{P_n\}$ satisfies equation (1).

Conversely, suppose the Appell set $\{P_n\}$ satisfies equation (1) (with $\lambda_0 = 0$); we shall show that L is defined by (19) for suitably chosen numbers $\{l_{kk}\}$. *Define* $\{l_{kk}\}$ by (3), and with this choice of $\{l_{kk}\}$, define an operator L^* by (19):

$$L^*(x; t) \sim \sum_{k=1}^{\infty} l_{kk} L_{[k]}(x; t),$$

where the function $A(t)$ of (15) is the generating function for $\{P_n\}$. From the half of the theorem already established, $\{P_n\}$ satisfies the equation $L^*[y] = \lambda^* y$, the set $\{\lambda_n^*\}$ being given by (3). But $\{\lambda_n\}$ also is given by (3), so that $\lambda_n^* = \lambda_n$. Subtracting the two equations, we find

$$(L - L^*)[P_n] = 0, \qquad (n = 0, 1, \cdots).$$

If we set $n = 0, 1, \cdots$ successively, we find that

$$L_0(x) - L_0^*(x) = L_1(x) - L_1^*(x) = \cdots = L_n(x) - L_n^*(x) = \cdots = 0.$$

That is, $L^* \equiv L$. This proves the converse.

We next determine the condition that an Appell set satisfy a finite order equation of type (1). Two well known examples are the polynomial sets $\{(x-a)^n/n!\}$ and $\{H_n(x)\}$, the latter being

the Hermite polynomials. They satisfy the following equations, respectively:

$$L[y(x)] \equiv (x - a)y'(x) = \lambda y(x), \qquad (\lambda_n = n);$$

$$L[y(x)] \equiv 2xy'(x) - y''(x) = \lambda y(x), \qquad (\lambda_n = 2n).$$

The corresponding generating functions are respectively $A(t) = e^{-\alpha t}$ and $A(t) = e^{-t^2/4}$.

THEOREM 4. *A necessary and sufficient condition that an Appell set* $\{P_n\}$, *with generating function* $A(t)$, *satisfy a finite order equation of type* (1) *is that* $A(t)$ *have the form*

$$(20) \qquad\qquad A(t) = e^{Q(t)},$$

where $Q(t)$ *is a polynomial.*

Suppose that $A(t)$ has the form (20), where Q is of degree k. Then from (9), $B(t) = Q'(t)$, a polynomial of degree $k-1$, so that $B(t) = b_0 + b_1 t + \cdots + b_{k-1} t^{k-1}$. The universal equation (10) for Appell polynomial sets therefore reduces to

$$L[P_n(x)] = (b_0 + x)P_n'(x) + b_1 P_n''(x) + \cdots + b_{k-1} P_n^{(k)}(x)$$
$$= nP_n(x), \qquad\qquad (n = 0, 1, \cdots),$$

and this is an equation of finite order (namely, order k).

Conversely, suppose the Appell set $\{P_n\}$ satisfies a finite order equation of type (1); we shall show that its generating function $A(t)$ has the form (20). Let the equation be

$$L[P_n(x)] \equiv L_0(x)P_n(x) + \cdots + L_k(x)P_n^{(k)}(x) = \lambda_n P_n(x),$$
$$(n = 0, 1, \cdots).$$

As already pointed out, we may suppose that $\lambda_0 = 0$ (so that $L_0(x) = 0$). If we write $L_n(x)$ in the form (2), then λ_n is given by (3), but since $L_n(x) \equiv 0$, $(n > k)$, λ_n reduces to the form (16). Let $L_{[k]}(x; t)$, $(k = 1, 2, \cdots)$, be the functions of (15), and define the operator L^* by $L^*(x; t) = \sum_{r=1}^{k} l_{rr} L_{[r]}(x; t)$, where $A(t)$ is the generating function for $\{P_n\}$. Then $\{P_n\}$ satisfies (by Theorem 2) the equation $L^*[P_n] = \lambda_n^* P_n$, where λ_n^* is determined by (16). Since (16) was used to determine λ_n, it follows

that $\lambda_n^* = \lambda_n$ for all n. Hence (as was argued before) we have $L^* \equiv L$. That is, the given equation for $\{P_n\}$ is defined by

(a) $$L(x; t) = \sum_{r=1}^{k} l_{rr} L_{[r]}(x; t).$$

But to begin with we had

(b) $$L(x; t) = \sum_{r=1}^{k} L_r(x) t^r.$$

Hence

(c) $$\sum_{r=1}^{k} l_{rr} L_{[r]}(x; t) = \sum_{r=1}^{k} L_r(x) t^r.$$

Let

(d) $$\frac{A^{(r)}(t)}{A(t)} = a_{r0} + a_{r1} t + \cdots, \qquad (r = 0, 1, \cdots, k).$$

Expanding the left member of (c) in a power series in t, and observing (from the right member) that powers of t higher than the kth degree cannot occur, we obtain the equations

(e) $$l_{11}(a_{1n}) + l_{22}(a_{2,n-1} + 2xa_{1,n-1}) + \cdots$$
$$+ l_{kk}\left(a_{k,n-k+1} + \frac{k}{1!} xa_{k-1,n-k+1} + \cdots\right.$$
$$\left. + \frac{k(k-1)\cdots 2}{(k-1)!} x^{k-1}a_{1,n-k+1}\right) = 0, \quad (n \geq k).$$

These relations are to hold identically in x. Of the numbers $1, \cdots, k$, let s be the largest for which $l_{ss} \neq 0$. It then follows from (e) that

(f) $$a_{1,n-s+1} = 0, \qquad\qquad (n \geq k).$$

That is,

$$\frac{A'(t)}{A(t)} = B(t) = a_{10} + a_{11}t + \cdots + a_{1,k-s} t^{k-s},$$

so that (20) holds, where $Q(t)$ is a polynomial of degree $k-s+1$. This completes the proof.

If the Appell set $\{P_n\}$ satisfies an equation of type (1) of finite order, then there are infinitely many other equations (of type (1)) of finite order for which the set is a solution. For example, from $L[P_n]=\lambda_n P_n$ follows $L^2[P_n]=\lambda_n^2 P_n$, $L^3[P_n]=\lambda_n^3 P_n$, $\cdots$. This raises the question: What is the minimum order of all equations satisfied by $\{P_n\}$? The answer is given by the following statement.

THEOREM 5. *Let the Appell set* $\{P_n\}$ *satisfy a finite order equation of type* (1), *so that its generating function* $A(t)$ *is given by* (20). *Then the minimum order of all equations* (*of type* (1)) *satisfied by* $\{P_n\}$ *is precisely the degree*† *of the polynomial* $Q(t)$.

Let $Q(t)$ be of degree m. That the minimum order k cannot exceed m is shown by the proof of the first half of Theorem 4, so that $k \leq m$. Let‡ $L[P_n]=\lambda_n P_n$ be the equation of minimum order k. From the proof of the second part of Theorem 4, we see that there is an integer s for which $1 \leq s \leq k$, and such that $Q(t)$ is of degree $k-s+1$. That is, $m=k-s+1$. But $k \leq m$, so that $k \leq k-s+1$. This requires that $s=1$, from which follows the desired conclusion that $k=m$.

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† The single exception is when $Q(t)$ is of degree *zero*, in which case $A(t)$ is a constant and $P_n(x)=cx^n/n!$. This set satisfies a first order equation.

‡ As usual it is no restriction to suppose, as we do, that $\lambda_0=0$.