THE METHOD OF MOMENT DISTRIBUTION FOR THE ANALYSIS OF CONTINUOUS STRUCTURES

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1. *Introduction.* If ab is any member of constant cross-section forming part of a continuous structure, the moments at its ends, M_{ab} and M_{ba}, are given in terms of the angle-changes θ_a, θ_b at its ends and the lateral deflection per unit of length R by the slope-deflection equations*

$$M_{ab} = 2EK_{ab}(2\theta_a + \theta_b - 3R) \mp C_{ab},$$
$$M_{ba} = 2EK_{ba}(2\theta_b + \theta_a - 3R) \pm C_{ba}.$$

Here $K_{ab} = K_{ba}$ denotes the sectional moment of inertia of ab divided by its length (I/l) and C_{ab}, C_{ba} are the *numerical* values of the fixed-end moments due to the loading on ab. For the derivation of these equations and the sign conventions employed, reference may be made to the Bulletin just cited.

When there is no lateral deflection, or when this is neglected, $R=0$, and we write the slope-deflection equations

$$M_{ab} = 2EK_{ab}(2\theta_a + \theta_b) + M_{ab}^{(0)}, \tag{1}$$
$$M_{ba} = 2EK_{ba}(2\theta_b + \theta_a) + M_{ba}^{(0)}, \tag{2}$$

in which $M_{ab}^{(0)}$, $M_{ba}^{(0)}$ denote the fixed-end moments *inclusive of sign.* Suppose also that at all joints of the structure, other than certain fixed ends, there is no external momental load. At any such joint a, we must have

$$\sum M_{ai} = 0, \tag{3}$$

the summation ranging over all members that meet at a. At a fixed end c, θ_c is given. Owing to the continuity of the structure, all members meeting at a joint a rotate through the same angle θ_a. At each such joint we have an equation of type (3). Thus we have precisely as many equations (3) as we have unknown angles, so that in general these equations determine the angles

* Bulletin No. 108, Engineering Experiment Station, University of Illinois, p. 20.

uniquely. Having found them, equations (1) and (2) give the terminal moments for each member of the structure.

Hardy Cross has devised a method of successive approximations for solving equations (1), (2), and (3) for the terminal moments M_{ab} that is very simple and effective. In his original paper* he regards the method as the mathematical parallel of a physical process. To quote Cross: "The beams are loaded or otherwise distorted while the joints are held against rotation; one joint is then allowed to rotate with the accompanying distribution of the unbalanced moment at that joint while the resulting moments are carried over to the adjacent joints; then another joint is allowed to rotate while the others are held against rotation; and the process is repeated until the joints are 'eased down' into equilibrium."

2. *The Successive Approximations.* To begin the calculation corresponding to this process we write at the ends of each member the fixed-end moments corresponding to the load it carries. Consider a joint a at which the members ab, ac, $\cdots$, ak meet. If the algebraic sum of the fixed-end moments at a,

$$S = M_{ab}^{(0)} + M_{ac}^{(0)} + \cdots + M_{ak}^{(0)},$$

is not zero, we add a moment $-S$ at a and distribute it among the members meeting there in the proportion of their stiffness. The *stiffness* of any member ab is defined as the moment which must be applied at its end a to make $\theta_a = 1$ when b is held fixed ($\theta_b = 0$). From (1) it is clear that the stiffness of ab is $4EK_{ab}$. Thus the stiffnesses of the members ab, ac, $\cdots$, ak are proportional to K_{ab}, K_{ac}, $\cdots$, K_{ak}. The balancing moment $-S$ is therefore distributed among these members so that they receive respectively

$$-\frac{K_{ab}}{\sum K_{ai}} \cdot S, \quad -\frac{K_{ac}}{\sum K_{ai}} S, \cdots, \quad -\frac{K_{ak}}{\sum K_{ai}} S.$$

While joint a is allowed to rotate all other joints are held fixed: in particular, $\theta_b = \theta_c = \cdots = \theta_k = 0$. Hence from equations

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* Proceedings of the American Society of Civil Engineers, May, 1930, pp. 919–928. For the paper with the complete discussions, see the Transactions of the American Society of Civil Engineers, vol. 96 (1932), pp. 1–156.

(1) and (2), a moment $4EK_{ab}\theta_a$ applied at end a of ab induces a moment $2EK_{ba}\theta_a$ at the fixed end b of ab. Thus, in the case of members of constant section, one-half of the moment distributed to ab in balancing the joint a is carried over to the end b. We therefore call 1/2 the *carry-over factor* for members of constant section.

The three basic operations involved in the above method of moment distribution are therefore:

BD (Balance and Distribute). *If the moments at a joint do not balance, add the balancing moment at the joint and distribute it to the members meeting there in the proportion of their stiffness.*

C (Carry-over). *One-half of each moment distributed to a member is carried over to its other end.*

The moments carried over in operation C destroy the balance achieved in operation BD. Hence BD must be repeated on the moments carried over. A succession of operations C and BD will be called a *cycle*. If, at the end of any cycle, the fixed-end moment at end a of member ab is added algebraically to all the distributed and carried-over moments there, the sum will give an approximate value of M_{ab}. We shall show that this approximate value approaches the true value of M_{ab} as the number of cycles performed increases indefinitely.

We next consider in detail this process of computing M_{ab}. The first cycle consists in writing the fixed-end moments at a and distributing the balancing moment at a according to operation BD. As regards the end a of ab, this gives the entries

$$M_{ab}^{(0)}$$

$$\text{(BD1)} \qquad M_{ab}^{(1)} = -\frac{K_{ab}}{\sum K_{ai}} \sum M_{ai}^{(0)} .$$

The moment $M_{ba}^{(1)}$, distributed to ab at b must now be carried over to a according to operation C. All these moments carried over to a create a new unbalance at this joint, which is then balanced and distributed according to operation BD. Thus the next two entries for the end a of ab are

$$\text{(C1)} \qquad \frac{1}{2} M_{ba}^{(1)} ,$$

(BD2) $$\frac{1}{2} M_{ab}^{(2)}, \quad \text{where} \quad M_{ab}^{(2)} = -\frac{K_{ab}}{\sum K_{ai}} \sum M_{ia}^{(1)}.$$

Similarly the next cycle of operations gives the entries at end a of ab:

(C2) $$\frac{1}{4} M_{ba}^{(2)},$$

(BD3) $$\frac{1}{4} M_{ab}^{(3)}, \quad \text{where} \quad M_{ab}^{(3)} = -\frac{K_{ab}}{\sum K_{ai}} \sum M_{ia}^{(2)}.$$

Continuing the cycles indefinitely and adding all the entries, we obtain

(4) $$\begin{aligned} M_{ab} = M_{ab}^{(0)} &+ \left[M_{ab}^{(1)} + \frac{1}{2} M_{ab}^{(2)} + \frac{1}{4} M_{ab}^{(3)} + \cdots \right] \\ &+ \frac{1}{2}\left[M_{ba}^{(1)} + \frac{1}{2} M_{ba}^{(2)} + \frac{1}{4} M_{ba}^{(3)} + \cdots \right]. \end{aligned}$$

If the end a of ab is fixed, equation (3) does not apply and hence the operation BD is omitted. In this case M_{ab} is built up from $M_{ab}^{(0)}$ and the successive moments carried over from b, and the first series in (4) is absent.

3. *Convergence of Series.* We shall first establish the convergence of the series in (4). Let C denote the sum of the absolute values of all the fixed-end moments for the entire structure. Then writing $p_{ab} = K_{ab}/\sum K_{ai}$, we have from (BD1)

$$\left| M_{ab}^{(1)} \right| \leqq p_{ab} \sum \left| M_{ai}^{(0)} \right| < C.$$

Now at any joint a, other than a fixed end,

$$\sum \left| M_{ai}^{(1)} \right| = (p_{ab} + p_{ac} + \cdots + p_{ak}) \left| \sum M_{ai}^{(0)} \right| \leqq \sum \left| M_{ai}^{(0)} \right|.$$

Forming such inequalities for all joints which are not fixed and adding the results we obtain

$$\sum \left| M_{ij}^{(1)} \right| \leqq \sum \left| M_{ij}^{(0)} \right| \leqq C.$$

Hence from (BD2)

$$\left| M_{ab}^{(2)} \right| \leqq p_{ab} \sum \left| M_{ia}^{(1)} \right| < C.$$

In the same way we may show that

$$| M_{ab}^{(n)} | < C, \qquad (n = 3, 4, \cdots).$$

Both series in (4) are therefore absolutely convergent.

4. *The Series Satisfy the Slope-Deflection Equations.* If we identify the first series in (4) with $4EK_{ab}\theta_a$, then the second series must represent $2EK_{ba}\theta_b$, and (4) reduces to equation (1). That this identification is justified is readily seen from the fact that after any operation BD, the summed moments at a joint a (which is not fixed) satisfy equation (3); for the operation BD was expressly designed to accomplish this. Hence in the limit the moments M_{ab} given by (4) satisfy the system of equations of type (3)—one equation for each free joint. Hence we see that the sums of the convergent series

$$M_{ij}^{(1)} + \frac{1}{2} M_{ij}^{(2)} + \frac{1}{4} M_{ij}^{(3)} + \cdots, \qquad (i \text{ not a fixed joint}),$$

satisfy precisely the same system of linear equations that $4EK_{ij}\theta_i$ satisfy. Thus if the system of equations (3) has a unique solution for the angles θ_i, this solution is given by

$$(5) \qquad 4EK_{ij}\theta_i = M_{ij}^{(1)} + \frac{1}{2} M_{ij}^{(2)} + \frac{1}{4} M_{ij}^{(3)} + \cdots.$$

5. *Moment at a Fixed End.* If a is the fixed end of a member ab, the equation (3) does not apply and the operation BD is therefore omitted. The fixed-end moment $M_{ab}^{(0)}$ and the moments carried over from b will total up to the actual external moment acting on the structure at a. For on putting $\theta_a = 0$ in (1) we have

$$(6) \qquad M_{ab} = 2EK_{ab}\theta_b + M_{ab}^{(0)};$$

and from (4), on omitting the first series due to the operation BD and retaining the second due to operation C,

$$M_{ab} = M_{ab}^{(0)} + \frac{1}{2}\left[M_{ba}^{(1)} + \frac{1}{2} M_{ba}^{(2)} + \frac{1}{4} M_{ba}^{(3)} + \cdots \right].$$

In view of (5) the series in brackets sums to $4EK_{ba}\theta_b$, so that the limiting value of the right-hand side has precisely the value given in (6).

6. *Procedure for a Member Hinged at One End.* If a is the hinged end of a member ab, the end of a cycle will always reduce the total moment at a to zero. But this series of operations BD and C leading to the result zero may be avoided by taking the stiffness of ab proportional to $K'_{ab}=(3/4)K_{ab}$, and replacing the fixed-end moments $M_{ab}^{(0)}$, $M_{ba}^{(0)}$ at the outset by

$$H_{ab}^{(0)} = 0, \qquad H_{ba}^{(0)} = M_{ba}^{(0)} - \frac{1}{2} M_{ab}^{(0)}.$$

These are the terminal moments for ab due to the given loading when a is hinged and b fixed. No moments need then be carried over to a from b, and consequently there is nothing to carry back from a to b. To justify this procedure, we have from (1) and (2) in this case,

$$0 = 2EK_{ab}(2\theta_a + \theta_b) + M_{ab}^{(0)},$$
$$M_{ba} = 2EK_{ab}(2\theta_b + \theta_a) + M_{ba}^{(0)}.$$

On eliminating θ_a from these equations we obtain

$$\begin{aligned} M_{ba} &= 2EK_{ab}\frac{3}{2}\theta_b + M_{ba}^{(0)} - \frac{1}{2} M_{ab}^{(0)} \\ &= 4EK'_{ab}\theta_b + H_{ba}^{(0)}. \end{aligned}$$

But from the successive approximations for obtaining M_{ba}, with $H_{ba}^{(0)}$ as a starting moment and no carry-overs from a, we have

$$M_{ba} = H_{ba}^{(0)} + \left[M_{ba}^{(1)} + \frac{1}{2} M_{ba}^{(2)} + \frac{1}{4} M_{ba}^{(3)} + \cdots \right],$$

where, in view of (5), the series in brackets is precisely $4EK'_{ba}\theta_b$ as all distributions to ba are now made proportional to K'_{ba}.

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