# SOME MULTIPLICATION THEOREMS FOR THE NÖRLUND MEAN

BY FLORENCE M. MEARS

Absolute summability for the series $\sum_{n=1}^{\infty} u_n$ by the Cesàro mean and by the Riesz mean have been defined by Fekete* and by Obrechkoff,† respectively. In each case, theorems for the multiplication of series summed by these means have been proved.‡ The purpose of this paper is to establish a definition for absolute summability by the Nörlund mean, and to prove three multiplication theorems for this mean. Theorem 1 includes Mertens' theorem for convergent series and its extension for the Cesàro mean. Theorem 2 includes Cesàro's multiplication theorem. Theorem 3 includes the following theorem by M. J. Belinfante for the Cesàro mean.

*If $\sum_{n=1}^{\infty} u_n$ is summable $C_s$ to $U$, and if $\sum_{n=1}^{\infty} v_n$ is summable $C_r$ to $V$, and bounded $C_{r-1}$, $(s \geqq 0, r \geqq 1)$, the product series $\sum_{n=1}^{\infty} w_n$ is summable $C_{r+s}$ to $UV$.*§

For any given series $\sum_{k=1}^{\infty} u_k$, with terms real or complex, form the sequence $\{U_k\}$, where $U_k = \sum_{n=1}^{k} u_n$. Let $\{a_n\}$ be a sequence of positive numbers, and let $A_k = \sum_{n=1}^{k} a_n$. The series $\sum_{k=1}^{\infty} u_k$ is said to be summable to $U'$ by the Nörlund mean $A$ if

$$\lim_{n\to\infty} U_n' = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} a_{n-k+1} U_k}{A_n}$$

exists and is equal to $U'$.¶ If $\sum_{k=1}^{\infty} u_k'$, where $u_n' = U_n' - U_{n-1}'$, is absolutely convergent, we shall say that $\sum_{k=1}^{\infty} u_k$ is absolutely summable $A$. We shall assume that $\lim_{n\to\infty}(a_n/A_n) = 0$; then $A$ is a regular method of summation.‖

* Matematikai és Természettudományi Értesitö, vol. 32 (1914), pp. 389–425.

† Comptes Rendus, vol. 185 (1928), pp. 215–217.

‡ For discussion and references, see Kogbetliantz, Mémorial des Sciences Mathématiques, No. 51.

§ Koninklijke Akademie te Amsterdam, Verslag, vol. 32 (1923), pp. 177–189.

¶ Riesz, Proceedings of the London Mathematical Society, (2), vol. 22 (1923), p. 412.

‖ Riesz, loc. cit.

We shall consider also the series $\sum_{k=1}^{\infty} v_k$, and $\sum_{k=1}^{\infty} w_k$, the Cauchy product of series $\sum_{k=1}^{\infty} u_k$ and $\sum_{k=1}^{\infty} v_k$; we have the corresponding sequences $\{V_k\}$ and $\{W_k\}$.

We shall assume that we have also a regular Nörlund mean, $B$, defined by $\{b_k\}$, a sequence of positive numbers. We shall form the Nörlund means, $C$, defined by $\{c_n\} = \{\sum_{k=1}^{n} a_k b_{n-k+1}\}$, and $D$, defined by $\{d_n\} = \{\sum_{k=1}^{n} A_k b_{n-k+1}\}$.

THEOREM 1. *If $\sum_{k=1}^{\infty} u_k$ is summable $A$ to $U'$, and in addition, absolutely summable $A$, and if $\sum_{k=1}^{\infty} v_k$ is summable $B$ to $V'$, then $\sum_{k=1}^{\infty} w_k$ is summable $C$ to $U'V'$.*

PROOF. We shall prove the theorem first with the assumption that $V' = 0$. Let

$$U_n' = \frac{1}{A_n} \sum_{k=1}^{n} a_k U_{n-k+1}, \qquad V_n' = \frac{1}{B_n} \sum_{k=1}^{n} b_k V_{n-k+1},$$
$$W_n' = \frac{1}{C_n} \sum_{k=1}^{n} c_k W_{n-k+1};$$

let $\sum_{k=1}^{\infty} u_k'$, $\sum_{k=1}^{\infty} v_k'$, and $\sum_{k=1}^{\infty} w_k'$ be the corresponding series; then

$$| W_n' | \leqq \frac{1}{C_n} \Bigg\{ | u_1' | \left| \sum_{l=1}^{n} c_l V_{n-l+1} \right| + \sum_{k=2}^{n} \left[ | u_k' | \left| \sum_{l=1}^{n-k} V_l S + A_k b_1 V_{n-k+1} \right| \right] \Bigg\},$$

where

$$S = A_k b_{n-k-l+2} + \sum_{p=k+1}^{n-l+1} a_p b_{n-l-p+2}.$$

Hence

$$\begin{aligned} |W_n'| \leqq \frac{1}{C_n} \Bigg\{ & | u_1' | \left| \sum_{l=1}^{n} c_l V_{n-l+1} \right| \\ & + \sum_{k=2}^{n} \left[ | u_k' | A_k \left| \sum_{l=1}^{n-k+1} b_{n-k-l+2} V_l \right| \right] \\ & + \sum_{k=2}^{n-1} \left[ | u_k' | \left| \sum_{l=1}^{n-k} \sum_{p=k+1}^{n-l+1} a_p b_{n-l-p+2} V_l \right| \right] \Bigg\} \\ = P_n | u_1' | & + \sum_{k=2}^{n} [ | u_k' | Q_{nk} ] + \sum_{k=2}^{n-1} [ | u_k' | R_{nk} ]. \end{aligned}$$

Since $C$ includes $B$,*

$$\lim_{n\to\infty} P_n = 0. \tag{1}$$

For $2 \leqq k \leqq n$,

$$Q_{nk} < \frac{A_k \left| \sum_{l=1}^{n-k+1} b_{n-k-l+2} V_l \right|}{\sum_{l=k}^{n} A_l b_{n-l+1}}$$

$$< \frac{A_k \left| \sum_{l=1}^{n-k+1} b_{n-k-l+2} V_l \right|}{A_k \sum_{l=k}^{n} b_{n-l+1}} = \left| V'_{n-k+1} \right|.$$

Therefore

$$\sum_{k=2}^{n} Q_{nk} \left| u'_k \right| < \sum_{k=2}^{n} \left| V'_{n-k+1} \right| \left| u'_k \right|$$

$$= \sum_{k=2}^{\nu} \left| V'_{n-k+1} \right| \left| u'_k \right| + \sum_{k=\nu+1}^{n} \left| V'_{n-k+1} \right| \left| u'_k \right|,$$

where $\nu$ may be chosen so that $\nu$ and $n-\nu$ become infinite with $n$. Since $\lim_{n\to\infty} \left| V'_n \right| = 0$, for any $\epsilon$, $\nu$ and $n$ may be chosen sufficiently large so that

$$\sum_{k=2}^{n} \left[ Q_{nk} \left| u'_k \right| \right] < \epsilon. \tag{2}$$

For $2 \leqq k \leqq n-1$, we have

$$R_{nk} \leqq \frac{\sum_{l=1}^{n-2} a_{n-l+1} B_l \left| V'_l \right|}{C_n}$$

$$= \left[ \frac{1}{C_n} \sum_{l=1}^{n-2} a_{n-l+1} B_l \right] \left[ \sum_{p=1}^{n-2} \frac{a_{n-p+1} B_p \left| V'_p \right|}{\sum_{l=1}^{n-2} a_{n-l+1} B_l} \right]$$

$$< \sum_{p=1}^{n-2} \frac{a_{n-p+1} B_p \left| V'_p \right|}{\sum_{l=1}^{n-2} a_{n-l+1} B_l} = V''_{n-2},$$

* Nörlund, Lunds Universitet, Årsskrift, (2), vol. 16 (1919), No. 3.

where $V'_{n-2}$ is the $(n-2)$th term of the sequence obtained when $\{|V'_p|\}$ is summed by the matrix transformation $t_{np}$, where

$$t_{np} = \frac{a_{n-p+3}B_p}{\sum_{l=1}^{n} a_{n-l+3}B_l}$$

for $p \leqq n$, and $t_{np}=0$ for $p>n$.

This transformation is regular, since $A$ is regular; it follows that $\lim_{n\to\infty} V''_{n-2}=0$, and that for any $\epsilon$, we may find $n$ sufficiently large so that

$$(3) \qquad R_{nk} < \epsilon.$$

From (1), (2), (3), and the fact that $\sum_{k=1}^{\infty}|u'_k|$ converges, it follows that $\lim_{n\to\infty}|W'_n|=0$. This proves the theorem for $V'=0$.

If $V'\neq 0$, we consider the sequence $\{V_n - V'\}$; this sequence is summed by $B$ to 0. Hence the Cauchy product of $\sum_{n=1}^{\infty}u_n$ by $[v_1 - V'] + \sum_{n=2}^{\infty}v_n$ is summed by $C$ to 0; that is,

$$\lim_{n\to\infty}\left[ W'_n - V' \frac{\sum_{k=1}^{n} c_k U_{n-k+1}}{C_n} \right] = 0.$$

Since $C$ includes $A$,

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} c_k U_{n-k+1}}{C_n} = U';$$

therefore $\lim_{n\to\infty} W'_n = U'V'$.

THEOREM 2. *If $\sum_{k=1}^{\infty}u_k$ is summable $A$ to $U'$, and if $\sum_{k=1}^{\infty}v_k$ is summable $B$ to $V'$, then $\sum_{k=1}^{\infty}w_k$ is summable $D$ to $U'V'$.*

PROOF. We have

$$W'_n = \frac{1}{D_n}\sum_{k=1}^{n} d_k W_{n-k+1} = \sum_{k=1}^{n} g_{nk} U'_k V'_{n-k+1},$$

where $g_{nk} = A_k B_{n-k+1}/D_n$ for $k \leqq n$, and $g_{nk}=0$ for $k>n$.

Since $A$ and $B$ are regular, this method of summation is regular and $\lim_{n\to\infty} g_{n,n-k+1}=0$; it follows that*

$$\lim_{n\to\infty} \sum_{k=1}^{n} g_{nk} U_k' V_{n-k+1}' = U'V',$$

which completes the proof.

For the proof of Theorem 3 we require the following lemma.

LEMMA. *If* $\{X_n\}$ *and* $\{[\sum_{k=1}^{n} B_k y_{n-k+1}]/B_n\}$ *converge to* $X$ *and* $Y$, *respectively, and if* $\{[\sum_{k=1}^{n} b_k y_{n-k+1}]/b_n\}$ *is bounded, then*

$$\lim_{n\to\infty} \left\{ \sum_{k=1}^{n} a_{nk} X_k \frac{\sum_{l=1}^{n-k+1} b_l y_{n-k-l+2}}{b_{n-k+1}} \right\} = XY,$$

*provided that* (a) $\lim_{n\to\infty} a_{nk}=0$; (b) $\sum_{k=1}^{n} |a_{nk}| < M$ *for all* $n$, *where* $M$ *is a positive constant*; (c) $T'$ *includes* $T$, *where* $T'$ *and* $T$ *are triangular matrix transformations defined by* $t_{nk}' = a_{n,n-k+1}$ *and* $t_{nk}=b_k/B_n$.

PROOF. Let

$$\frac{\sum_{l=1}^{n-k+1} b_l y_{n-k-l+2}}{b_{n-k+1}} = y_{n-k+1}',$$

and let $a_{nk} y_{n-k+1}' = c_{nk}$. Let $Z_n = \sum_{k=1}^{n} c_{nk} X_k$. From (a), (b), and (c), it follows that $\lim_{n\to\infty} c_{nk}=0$, $\sum_{k=1}^{n} |c_{nk}| < M'$, where $M'$ is a positive constant, and $\lim_{n\to\infty} \sum_{k=1}^{n} c_{nk} = Y$. Choose $p$ such that for a given $\epsilon > 0$, $|X_k - X| < \epsilon/(2M')$ when $k > p$. For $k \leqq p$, $|X_k - X| < L$. Then

$$\left| Z_n - X\sum_{k=1}^{n} c_{nk} \right| \leqq \sum_{k=1}^{p} |c_{nk}| \, |X_k - X| + \sum_{k=p+1}^{n} |c_{nk}| \, |X_k - X|$$

$$\leqq L \sum_{k=1}^{p} |c_{nk}| + \frac{\epsilon}{2M'} |c_{nk}|.$$

Choose $N > p$, and such that $|c_{nk}| < \epsilon/(2pL)$ for $n > N$. Then for $n > N$, $|Z_n - X\sum_{k=1}^{n} c_{nk}| < \epsilon$, which proves the lemma.

---

* Dale, American Journal of Mathematics, vol. 47 (1925), p. 82.

THEOREM 3. *If* $\sum_{k=1}^{\infty} u_k$ *is summable A to U' and* $\sum_{k=1}^{\infty} v_k$ *summable B to V', and if*

$$\left| \frac{b_n v_1 + \cdots + b_1 v_n}{b_n} \right| < M,$$

*then* $\sum_{k=1}^{\infty} w_k$ *is summable C to U'V'.*

PROOF. Consider the triangular matrix definition

$$a_{nk} = \frac{A_k b_{n-k+1}}{A_1 b_n + \cdots + A_n b_1}.$$

This definition satisfies the three conditions of the lemma, for

$$\frac{A_k b_{n-k+1}}{\sum_{l=1}^{n} A_l b_{n-l+1}} < \frac{A_k b_{n-k+1}}{\sum_{l=k}^{n} A_l b_{n-l+1}} < \frac{b_{n-k+1}}{\sum_{l=k}^{n} b_{n-l+1}}; \tag{4}$$

$$\sum_{k=1}^{n} | a_{nk} | = 1; \tag{5}$$

$$A' = C'B', \tag{6}$$

where $A'$, $B'$, and $C'$ are triangular matrix definitions with

$$a'_{nk} = a_{n,n-k+1}, \qquad b'_{nk} = \frac{b_k}{B_n}, \qquad c'_{nk} = \frac{a_{n-k+1} B_k}{\sum_{l=1}^{n} A_l b_{n-l+1}}.$$

The definition $C'$ is regular. The theorem follows immediately from this lemma.

THE GEORGE WASHINGTON UNIVERSITY