THE HILBERT SPACE POSTULATES—A FURTHER REDUCTION

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In a recent issue of this Bulletin the writer gave a set of postulates for abstract Hilbert space in which equality was treated postulationally. The result there obtained was a set of eleven postulates, independent among themselves, defining a linear vector space of a special sort. Three more postulates were then added, to define a Hilbert space.* As a result of investigations carried on by I. E. Highberg and myself concerning postulate systems for general vector spaces,† it is now possible to make a further simplification of the Hilbert space postulates. The equality relation as an undefined idea of the system is not independent, and may be defined in terms of the other notions in the system. When this is done the number of postulates is reduced by three. The postulate system for Hilbert space is then the following, where A denotes either the complex number system C, or the real number system R.

1. *The class K contains at least one element.*
2. *If $x, y \in K$, then $x+y \in K$.*
3. *If $a \in A$ and $x \in K$, then $a \cdot x \in K$.*
4. *If $x, y \in K$, then $(x, y) \in A$.*
5. *If $x, y, z \in K$, then $(x+y, z) = (x, z)+(y, z)$.*
6. *If $x, y \in K$, then $(x, y) = \overline{(y, x)}$.*
7. *If $a \in A$ and $x, y \in K$, then $(a \cdot x, y) = a(x, y)$.*
8. *If $x \in K$, then $(x, x) \geqq 0$.*
9. *For each integer $n>0$ there exist elements $x_1, \cdots, x_n \in K$ such that $(a_1 \cdot x_1+a_2 \cdot x_2+ \cdots +a_n \cdot x_n, a_1 \cdot x_1+ \cdots +a_n \cdot x_n)=0$ if and only if $a_1=a_2= \cdots =a_n=0$.*
10. *K is separable according to the norm $(x, x)^{1/2} \equiv \|x\|$.*
11. *K is complete according to the norm $(x, x)^{1/2} \equiv \|x\|$.*

DEFINITION. *If $x, y \in K$, we say that x is equal to y and write $x=y$ if and only if $(x+-1 \cdot y, x+-1 \cdot y)=0$.*

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* A. E. Taylor, this Bulletin, vol. 41 (1935), pp. 439–448.

† A. E. Taylor and I. E. Highberg, *On postulate systems for normed vector spaces.*

In order to prove that these postulates do define a Hilbert space we must establish the following propositions, which were postulates in the original set.

I. *If* $x, y \in K$, *and if for each* $u \in K$, $(x+-1\cdot y)+u=u$, *then* $x=y$.

II. *If* $x \in K$ *and* $(x, x)=0$, *then* $x+y=y$ *for each* $y \in K$.

III. *If* $x, y, u \in K$ *and* $x=y$, *then* $(x, u)=(y, u)$.

PROOF OF I. Let

$$(((x + - 1\cdot y) + u) + - 1\cdot u, ((x + - 1\cdot y) + u) + - 1\cdot u) = 0.$$

Then, since u is arbitrary, we may replace it by $0\cdot u$. On expanding, with the aid of postulates 5, 6, 7, we easily find that $(x+-1\cdot y, x+-1\cdot y)=0$, or $x=y$.

PROOF OF II. Let $(x, x)=0$. Then, just as in the previous proof, we obtain the equation

$$((x + y) + - 1\cdot y, (x + y) + - 1\cdot y) = 0.$$

This means, by definition, $x+y=y$.

PROOF OF III. Let $(x+-1\cdot y, x+-1\cdot y)=0$, and let u be an arbitrary element of K. Then $(x, u)-(y, u)=(x+-1\cdot y, u)$. But by the Schwarz inequality, which does not depend for its proof on the use of proposition III, we have

$$0 \leqq |(x + - 1\cdot y, u)| \leqq (x + - 1\cdot y, x + - 1\cdot y)^{1/2}(u, u)^{1/2} = 0.$$

Therefore $(x, u)=(y, u)$.

Postulates 1-8 are independent among themselves, as is proved by the examples given in the previous paper. The independence of the set taken as a whole is an open question.

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