ON APPROXIMATION BY POLYNOMIALS TO A FUNCTION ANALYTIC IN A SIMPLY CONNECTED REGION*

BY O. J. FARRELL

In a previous paper† the writer studied expansions in series of polynomials of a function $f(z)$ analytic in a limited simply connected region G where $f(z)$ is known either to be bounded in G or such that the double integral over G of the pth power $(p>0)$ of the modulus of $f(z)$ exists.‡ The present note contains an extension of each of the two theorems obtained in the earlier paper. The extended theorems now read as follows.

THEOREM A. *Let G be a limited simply connected region of the z plane. Then in order that corresponding to every function $f(z)$ analytic and bounded in G there shall exist a sequence of polynomials $\{p_n(z)\}$ which converge to $f(z)$ in G as $n\to\infty$ and at the same time such that*

$$\overline{\lim_{n\to\infty}}\ [\,|p_n(z)|,\ z \text{ in } G] \leqq \overline{\text{bound}}\ [\,|f(z)|,\ z \text{ in } G], \tag{1}$$

it is necessary and sufficient that the boundary of G be also the boundary of an infinite region.

THEOREM B. *In the z plane let G be a limited simply connected region whose boundary is also the boundary of an infinite region. Let $f(z)$ be analytic in G and such that*

$$\iint_G |f(z)|^p dS, \qquad (p>0), \tag{2}$$

exists. Then there exists a sequence of polynomials $\{p_n(z)\}$ such that

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† This Bulletin, vol. 40 (1934), pp. 908–914.

‡ The writer is indebted to Professor J. L. Walsh for having suggested a study of these two problems and also to Professor Torsten Carleman for sending a reprint of his paper on approximation to analytic functions by linear aggregates of prescribed powers (Arkiv för Matematik, Astronomi och Fysik, vol. 17 (1923), pp. 1–30).

$$\lim_{n\to\infty} \iint_G |f(z) - p_n(z)|^p dS = 0. \tag{3}$$

It will be noticed that in the first of these theorems we no longer say that there exist polynomials $\{p_n(z)\}$ which converge to $f(z)$ continuously in G, but merely that there exist polynomials $\{p_n(z)\}$ which converge to $f(z)$ in G. For if polynomials $\{p_n(z)\}$ converge to $f(z)$ in G so that (1) holds, these polynomials are uniformly bounded in G and thus form in G a normal family of analytic functions from which can be chosen a subsequence converging to $f(z)$ continuously in G. Hence, whenever there exists a sequence $\{p_n(z)\}$ converging to $f(z)$ in G so that (1) holds, there exists also a sequence which fulfills (1) and converges to $f(z)$ continuously in G. It will be seen too that in the second theorem we no longer say that there exist polynomials $\{p_n(z)\}$ which converge to $f(z)$ continuously in G and for which (3) holds, but merely that there exist polynomials $\{p_n(z)\}$ for which (3) holds. This is because we have since found in the literature a lemma by Walsh* giving assurance that if (3) holds, the polynomials $\{p_n(z)\}$ do converge to $f(z)$ continuously in G, so that specific mention of the convergence may be omitted.

The proof of Theorems A and B requires only a slight modification of the proof of the two corresponding theorems in the previous paper. This modification is brought about by observing that if G is any limited simply connected region whose boundary also bounds an infinite region, then there exists a sequence of regions $\{G_n\}$, each of which is a Jordan region lying interior to its predecessor and which are all such that the sequence $\{G_n\}$ converges to G as kernel.† If we use such a sequence of regions $\{G_n\}$, the proofs of Theorems 1 and 2 of the previous paper apply to Theorems A and B, respectively, of the present note. It is to be remarked, however, that uniform approximation to $f_n(z)$ or $F_n(z)$ in $\overline{G}$ by a polynomial with arbitrarily small error does not now follow directly from the theorem of Walsh that was

* Transactions of this Society, vol. 33 (1931), pp. 370–388, Lemma on p. 387.

† Compare Carathéodory, Mathematische Annalen, vol. 72 (1912), pp. 107–144, Chapter 3; or Walsh, Transactions of this Society, vol. 32 (1930), pp. 335–390, proof of Theorem X.

used before, but does follow indirectly from it, since $f_n(z)$ or $F_n(z)$, being analytic in the closed Jordan region $\overline{G}_{n+1}$, can be uniformly approximated with arbitrarily small error in $\overline{G}_{n+1}$ by a polynomial, and hence can be so approximated in $\overline{G}$. Indeed, Runge's classical theorem on polynomial approximation could be applied here and for that matter could have been used in the previous paper. For the closed region $\overline{G}$ is interior to every region G_n and hence the function $f_n(z)$ or $F_n(z)$ can be approximated as closely as desired in $\overline{G}$ by a polynomial in z.

The proof of Theorem B and the proof of the sufficiency of the condition of Theorem A are the same from this point on as for the corresponding theorems in the earlier paper. And the proof of the necessity of the condition of Theorem A is also contained there in §5 as "A Remark on Theorem 1."

The writer hopes in a forthcoming paper to be able to determine the most general type of limited simply connected region to which Theorem B can be extended. That *this theorem does not hold for an arbitrary finite simply connected region* is shown by the following simple example.

Let G be taken as the region bounded by the two circles $|z|=a$, $|z|=b$, $b>a$, and by the line segment $a\leqq z\leqq b$. Let $f(z)=1/z$. Denote by Q the doubly connected region bounded by the two circles. If now there existed a positive number p together with a sequence of polynomials $\{p_n(z)\}$ such that

$$\lim_{n\to\infty}\iint_G |p_n(z)-1/z|^p dS = 0,$$

it would follow that

$$\lim_{n\to\infty}\iint_Q |p_n(z)-1/z|^p dS = 0. \tag{4}$$

Consequently the polynomials $\{p_n(z)\}$ would converge* to $1/z$ in Q and the convergence would be uniform on any closed point set lying in Q, say on a circle $|z|=c$, $a<c<b$. Hence, the polynomials $\{p_n(z)\}$ would converge uniformly on and within the circle $|z|=c$ to a limit function analytic within this circle. But

* This convergence would follow by the lemma of Walsh to which reference was made above in the paragraph immediately following the statement of Theorem B.

such a function could not be equal to $1/z$ in the ring region between the circles $|z| = a$ and $|z| = c$.

The main feature of this example is the use of the lemma of Walsh whereby we know that if in a finite region R we have a sequence of polynomials $\{p_n(z)\}$ for which

$$\lim_{n\to\infty} \iint_R |p_n(z) - f(z)|^p dS = 0, \qquad (p > 0),$$

where $f(z)$ is a given function analytic in R, then the polynomials $\{p_n(z)\}$ converge to $f(z)$ continuously in R. The converse is not always true, as was shown by an example in §4 of our previous paper. There is, however, a qualified form of converse which does hold for an arbitrary limited region and for an arbitrary function analytic therein. We may state this result as follows.

THEOREM C. *Let R be a limited region of the z plane, and let $f(z)$ be analytic in R and such that*

$$\iint_R |f(z)|^p dS, \qquad (p > 0),$$

exists. If polynomials $\{p_n(z)\}$ exist which converge to $f(z)$ continuously in R and for which

$$\lim_{n\to\infty} \iint_R |p_n(z)|^p dS = \iint_R |f(z)|^p dS,$$

then

$$\lim_{n\to\infty} \iint_R |f(z) - p_n(z)|^p dS = 0.$$

The proof of this theorem is already contained in the latter part of the proof of Theorem 2 in our previous paper.

We close this note with a result closely connected with Theorem A.

THEOREM D. *Let G denote a limited simply connected region whose boundary does not bound an infinite region. Let B denote the boundary of the infinite region among the regions into which the closed region $\overline{G}$ separates the plane, and let Γ denote the region (also simply connected) consisting of all the points which can be joined to an arbitrary fixed point of G by a Jordan arc containing* *no point of B. Let $f(z)$ be analytic and bounded in G. A necessary and sufficient condition for the existence of polynomials $\{p_n(z)\}$ which converge to $f(z)$ in G so that* (1) *holds is that there exist a function analytic and bounded in Γ and equal to $f(z)$ in G.*

The proof of this theorem is much the same as for Theorem A taken together with the remark of §5 in the earlier paper and is therefore omitted.

The conclusion of Theorem D simply means of course that $f(z)$ shall be analytically extensible throughout Γ.

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A GENERALIZED PARSEVAL'S RELATION

BY E. S. QUADE

A function $\phi(x)$ which is non-negative, convex, and satisfies the conditions $\phi(0) = 0$ and $(\phi(x)/x) \to \infty$ as $x \to \infty$ will be called a Young's function. Given such a function $\phi(x)$, a second function, $\psi(x)$, with the same properties can be found such that Young's inequality, $ab \leqq \phi(a) + \psi(b)$, holds for every $a, b \geqq 0$. The functions $\phi(x)$ and $\psi(x)$ are then said to be complementary in the sense of Young.†

If $x(t)$ is such that $\int_a^b \phi(|x|)dt$ exists, $x(t)$ is said to belong to the space $L_\phi(a, b)$. This space is not necessarily linear.‡ For this reason we denote by $L_\phi^*(a, b)$ the class of all functions $x(t)$, $a \leqq t \leqq b$, such that the product $x(t)y(t)$ is integrable for every $y(t) \epsilon L_\psi(a, b)$. If we put

$$\|x\|_\phi = \sup_y \left| \int_a^b x(t)y(t)dt \right|$$

for all $y(t)$ with

$$\rho_y \equiv \int_a^b \psi(|y|)dt \leqq 1,$$

then L_ϕ^* is a linear metric, and complete space.§ A function

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† W. H. Young, Proceedings Royal Society, (A), vol. 87 (1912), pp. 225–229.

‡ W. Orlicz, *Über eine gewisse Klasse von Räumen vom Typus B*, Bulletin, Académie Polonaise, (A), Cracovie (1932).

§ A. Zygmund, *Trigonometrical Series*, 1935, pp. 95–100.