the connected point sets B_n and B_m have no point in common, there clearly exists no point set consisting of a finite number of connected subsets of M and separating G from A in M.

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A THEOREM ON PLANE CONTINUA*

BY W. T. REID

1. *Introduction.* In this paper the following theorem is proved.

THEOREM. *If M is a plane continuum, and K is a proper subcontinuum of M, then at least one component of $M-K$ has a limit point in K.*

Two points sets are *mutually separated* if they are mutually exclusive and neither of them contains a limit point of the other. A point set is said to be *connected* if it is not the sum of two non-vacuous mutually separated point sets. A point set which is both connected and closed is a *continuum.* A *component* of a point set N is a connected subset of N which is not a proper subset of any other connected subset of N. The set of all points in the plane will be denoted by S.†

2. *Proof of the Theorem.* If M is a bounded continuum and K is a proper subcontinuum of M, it is well known that every component of $M-K$ has a limit point in K.‡ If M is unbounded then it is no longer true that *every* component of $M-K$ has a limit point in K.§

If K is a bounded subcontinuum of an unbounded plane continuum M, then the above theorem may be proved readily. For

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* Presented to the Society, December 28, 1934. The result of this paper was obtained in 1928, while the author was a student under R. L. Moore at the University of Texas. Recently both R. L. Moore and J. H. Roberts have proved results beyond that of the present paper and have suggested that I publish my original result.

† These definitions are those customarily used in point set theory. See, for example, R. L. Moore, *Foundations of Point Set Theory*, Colloquium Publications of this Society, vol. 13. For brevity, this treatise will be referred to as "Moore."

‡ See, for example, Moore, p. 24.

§ See Moore, p. 25, example 2.

let J be a simple closed curve enclosing K. If D is the interior of J and M_1 denotes the component containing K of the subset of M belonging to $D+J$, then M_1 is a bounded continuum containing K as a proper subcontinuum. As indicated above, if C_1 is a component of M_1-K, then C_1 has a limit point in K. Hence the component C of $M-K$ which contains C_1 has a limit point in K, and the theorem is proved.

Now suppose that K is an unbounded proper subcontinuum of the plane continuum M. If x is a point of $M-K$, the component of $M-K$ containing x will be denoted by C_x. On the assumption that the above theorem is false, it is seen that each component C_x is an unbounded continuum. It follows, therefore, that M is the sum of a set of mutually exclusive unbounded continua consisting of K and the totality of components of $M-K$. This set of mutually exclusive continua is necessarily uncountable in number.*

We shall prove the preceding theorem by showing that the assumption that it is false leads to a contradiction. The proof depends upon the following auxiliary lemma which will be established in the next section.

LEMMA 1. *On the assumption that the above theorem is false, the components C_x of $M-K$ satisfy the following conditions*: (α) *for each component C_x, $M-C_x$ is connected*; (β) *if x_1 and x_2 are two points of $M-K$ such that C_{x_1} and C_{x_2} are mutually exclusive, then there does not exist a simple continuous arc x_1x_2 such that $x_1x_2-(x_1+x_2)$ belongs to $S-M$.*

With the help of this lemma the proof is as follows. Let D be a complementary domain of K containing points of $M-K$. Since M is a continuum, it follows that D must contain infinitely many of the components C_x of $M-K$. Let x_1 and x_2 be points of $M-K$ belonging to D such that C_{x_1} and C_{x_2} are mutually exclusive, and consider a simple continuous arc x_1x_2 which belongs to D.† Let X denote the subset of points x of $M-K$ such that

* See R. L. Moore, *Concerning the sum of a countable number of mutually exclusive continua in the plane*, Fundamenta Mathematicae, vol. 6 (1924), pp. 189–202.

† For a proof that such an arc exists, see R. L. Moore, *On the foundations of plane analysis situs*, Transactions of this Society, vol. 17 (1916), pp. 131–164, in particular, p. 137; see also, Moore, p. 86.

the component C_x has a point in common with the arc x_1x_2. Now if X were connected, it would follow that C_{x_1} and C_{x_2} belong to a single component of $M-K$, which is a contradiction. Hence X is the sum of two mutually separated point sets X_1 and X_2. Clearly, if x is a point of $M-K$ belonging to a set X_i, $(i=1, 2)$, then every point of C_x belongs to the same X_i. Let Y_i, $(i=1, 2)$, denote the set of points common to X_i and the arc x_1x_2. The sets Y_i are seen to be closed and mutually exclusive. There exists, therefore, a subarc $x_{10}x_{20}$ of x_1x_2 such that x_{i0} belongs to Y_i and $x_{10}x_{20}-(x_{10}+x_{20})$ belongs to x_1x_2-X and therefore to $S-M$. This, however, is a contradiction to conclusion (β) of Lemma 1, since x_{10} and x_{20} belong to mutually exclusive components of $M-K$. We have proved that the assumption that our theorem is false leads to a contradiction. Hence the theorem is true.

3. *Proof of Lemma* 1. Conclusion (α) of Lemma 1 has been established by Knaster and Kuratowski.* We shall prove conclusion (β) by showing that the assumption that it is false leads to a contradiction. For suppose that there are two points x_1 and x_2 of $M-K$ such that C_{x_1} and C_{x_2} are mutually exclusive and there exists a simple continuous arc x_1x_2 such that $x_1x_2-(x_1+x_2)$ belongs to $S-M$. Let D denote the complementary domain of the continuum K which contains $C_{x_1}+C_{x_2}+x_1x_2$, and denote by M_1 the subset of M belonging to D. Since M is a continuum, M_1 is seen to contain infinitely many of the components C_x of $M-K$. Now let x' be a point of $M_1-(C_{x_1}+C_{x_2})$, and P a point of the arc x_1x_2 between x_1 and x_2. Since x' and P are both points of D, they are not separated by the continuum K. Moreover, since $x_1x_2-(x_1+x_2)$ belongs to $S-M$ and $M-C_{x_i}$, $(i=1, 2)$, is connected in view of conclusion (α), there exists a complementary domain D_i of C_{x_i} containing the points P and x', and hence the continuum C_{x_i}, $(i=1, 2)$, does not separate x' and P. It follows, therefore, that $K+C_{x_1}+C_{x_2}$ does not separate† x' and P, and hence there exists a simple continuous arc $x'P$ belong-

* See B. Knaster and C. Kuratowski, *Sur les ensembles connexes*, Fundamenta Mathematicae, vol. 2 (1921), pp. 206–255, in particular, p. 214, Theorem X.

† See B. Knaster and C. Kuratowski, *Sur les continus non-bornés*, Fundamenta Mathematicae, vol. 5 (1924), pp. 23–58, in particular, p. 35.

ing to $S-(K+C_{x_1}+C_{x_2})$. Let O denote the first point of the arc $x'P$ belonging to the arc x_1x_2, and x_3 the last point of the arc $x'O$ belonging to M. The point O is seen to be between x_1 and x_2 on the arc x_1x_2, and x_3 is a point of $M-K$ such that C_{x_3} is distinct from C_{x_1} and C_{x_2}.

Now let H_j, $(j=1, 2, 3)$, denote the continuum consisting of the component C_{x_j} and the arc Ox_j. Each of the three unbounded continua H_j contains the point O, no two of them have in common any point except O, and no one of them is separated by the omission of O. It then follows* that there is no complementary domain of the continuum $H=H_1+H_2+H_3$ whose boundary contains a point, distinct from O, of each of the three continua H_j. If $\mathcal{D}$ denotes the complementary domain of H containing the continuum K, the boundary of $\mathcal{D}$, aside from the point O, belongs to some two of the continua H_j. For definiteness, suppose that the boundary of $\mathcal{D}$ is a subset of H_1+H_2. Let $\mathcal{D}_0$ denote the complementary domain of the continuum H_1+H_2 containing the continuum C_{x_3}, and denote by M_0 the subset of M belonging to $\mathcal{D}_0$. Since $\mathcal{D}$ is a complementary domain of H, the domain $\mathcal{D}_0$ is distinct from the domain $\mathcal{D}$ and the point set M_0 does not contain K. Moreover, since M is a continuum, infinitely many of the components C_x belong to M_0.

Now consider the closed point set $N=M_0+C_{x_1}+C_{x_2}$. If N is connected, we have a contradiction to the assumption that C_{x_j}, $(j=1, 2, 3)$, is a component of $M-K$. Suppose, on the other hand, that N is not connected. It will now be shown that under this hypothesis the point set N is the sum of two mutually exclusive closed point sets N_1 and N_2 such that N_1 contains both C_{x_1} and C_{x_2}. Since N is closed, the assumption that N is not connected implies that $N=N_{10}+N_{20}$, where N_{10} and N_{20} are mutually exclusive closed point sets. It may be supposed without

* For let A be a point of $S-H$ and subject the plane to an inversion about A. If $\bar{O}$, $\bar{H}_j$, $(j=1, 2, 3)$, denote the images of O, H_j, $(j=1, 2, 3)$, under this inversion, then the bounded continua $L_j=\bar{H}_j+A$ satisfy the following conditions: each of the continua L_j contains the distinct points A and $\bar{O}$, no two of them have in common any point except A and $\bar{O}$, and no one of them is separated by the omission of $A+\bar{O}$. It then follows [see, for example, Moore, p. 291] that there is no complementary domain of the continuum $L=L_1+L_2+L_3$ whose boundary contains a point, distinct from A and $\bar{O}$, of each of the three continua L_j, $(j=1, 2, 3)$. Since each complementary domain of L is the image of a complementary domain of H, the proof of the above stated result is complete.

loss of generality that N_{10} contains more than one of the components C_x of $M-K$ which belongs to N. If N_{10} contains both C_{x_1} and C_{x_2}, the property stated above is true for $N_1=N_{10}$, $N_2=N_{20}$; if N_{10} contains neither C_{x_1} nor C_{x_2}, the property is true for $N_1=N_{20}$, $N_2=N_{10}$. Finally, suppose that (i, j) is a permutation of (1, 2) such that N_{10} contains C_{x_i} and N_{20} contains C_{x_j}. Since N_{10} contains components of $M-K$ distinct from C_{x_i}, the closed point set N_{10} is not connected. Therefore, $N_{10}=N_{101}+N_{102}$, where N_{101} and N_{102} are mutually exclusive closed point sets and N_{101} contains C_{x_i}. The above stated property is then true for $N_1=N_{101}+N_{20}$, $N_2=N_{102}$. Now since N_1 contains both C_{x_1} and C_{x_2}, we see that N_2 is a subset of M_0 and hence contains no limit point of $M-N$. Consequently, N_2 and $M-N_2$ are mutually separated, which is a contradiction to the assumption that M is a continuum.

On the assumption that conclusion (β) of Lemma 1 is false we have thus been led to a contradiction. Hence this conclusion is true, and Lemma 1 is established.

4. *Remarks*. In a letter to me, J. H. Roberts has stated that he has discovered an example of a continuum in three-dimensional euclidean space showing that the result of the theorem of this paper does not hold when the condition that the continuum M be a plane continuum is omitted. There still remains the interesting question as to whether or not the statement obtained by replacing in our theorem the word "component" by "maximal strongly connected subset"* is true. That this latter question may also be answered in the negative when the condition that the continuum M be a plane continuum is omitted is a consequence of an example given by Knaster and Kuratowski of a three-dimensional indecomposable continuum each of whose composants is a continuum.†

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* A point set N is said to be *strongly connected* if for every two points x and y of N there exists a continuum which contains both x and y and which is a subset of N. A maximal strongly connected subset of N is a strongly connected subset of N which is not a proper subset of any other strongly connected subset of N.

† See B. Knaster and C. Kuratowski, *Sur les continus non-bornés*, Fundamenta Mathematicae, vol. 5 (1924), pp. 23–58, §4.