ON A THEOREM OF PLESSNER

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Plessner‡ has shown that if $f(x) \subset L_2$ on $(-\pi, \pi)$ and

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

then

$$\sum_{n=2}^{\infty} (a_n \cos nx + b_n \sin nx)(\log n)^{-1/2} \tag{1}$$

converges almost everywhere on $(-\pi, \pi)$. We designate the set where (1) converges by $E(Pl, f)$. This set is then known to be of measure 2π. The sets $E(F, f)$, consisting of the points where

$$\phi(t) = f(x + t) + f(x - t) - 2f(x) \to 0, \text{ as } t \to 0,$$

and $E(L, f)$, consisting of the points where

$$\Phi(t) = \int_0^t |\phi(\tau)| \, d\tau = o(t), \text{ as } t \to 0,$$

are of much importance in the theory of Fourier series. The set $E(L, f)$ is known to be of measure 2π for all integrable functions. It is obvious that

$$E(F, t) \subset E(L, f).$$

We propose in this note to investigate the inclusion relationships between these sets and $E(Pl, f)$. We shall prove

$$E(F, f) \not\subset E(Pl, f), \tag{2}$$

and

$$E(Pl, f) \not\subset E(L, f). \tag{3}$$

We first consider (2). Plessner§ showed that, if (1) converges,

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‡ A. Plessner, Journal für Mathematik, vol. 155 (1926), pp. 15–25.

§ Loc. cit., p. 22.

(4) $S_n(x) = \frac{a_0}{2} + \sum_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x) = o\{(\log n)^{1/2}\}.$

However, it is well known† that, for a continuous function, the estimate

(5) $$S_n(x) = o(\log n)$$

cannot be improved. This implies that (4) need not be satisfied at every point of continuity and hence

$$E(F, f) \not\subset E(Pl, f).$$

In order to prove (3) we shall construct a function $f(x) \subset L_2$ on $(-\pi, \pi)$ for which (1) converges at $x = 0$ but such that

(6) $$\int_0^t |\phi(\tau)|\, d\tau \neq o(t) \text{ as } t \to 0.$$

The function is similar to one constructed by Paley‡ for another purpose. We define $f(x)$ by

$$f(x) = \begin{cases} x\{(x - n^{-1})n \log n\}^{-1}, \text{ if } n^{-2} \geqq |x - n^{-1}| \geqq n^{-3}, \\ \qquad\qquad\qquad\qquad (n = 3, 4, \cdots), \\ 0, \text{ elsewhere on } (0, \pi), \\ f(-x) \text{ for } 0 \geqq x \geqq -\pi. \end{cases}$$

Then, since

$$\int_0^\pi |f(x)|^2 dx = O\left\{\sum_{n=3}^{\infty} n^{-4}(\log n)^{-2} \int_{n^{-3}}^{n^{-2}} \frac{dx}{x^2}\right\}$$
$$= O\left\{\sum_{n=1}^{\infty} n^{-1}(\log n)^{-2}\right\},$$

$f(x) \subset L_2$ on $(-\pi, \pi)$. We have at $x=0$, $\phi(t) = 2f(t)$ and

$$\int_0^t |\phi(t)|\, dt > \sum_{n=[1/t]+1}^{\infty} (n^2 \log n)^{-1} \int_{n^{-2}}^{n^{-3}} \frac{dx}{x} = \sum_{n=[1/t]+1}^{\infty} n^{-2} > \frac{t}{3},$$

† P. Du Bois Reymond, Abhandlungen der Bayerische Akademie, vol. 12, part 2 (1876). There is a simplification of Du Bois Reymond's method given by Lebesgue, *Leçons sur les Séries Trigonométriques*, 1906, pp. 84–86.

‡ R. E. A. C. Paley, Proceedings Cambridge Philosophical Society, vol. 26 (1930), pp. 173–203; see §10, pp. 201–203.

for t sufficiently small. Now we consider $S_m(0)$. It is well known that

$$S_m(0) = \frac{1}{2\pi}\int_0^{\pi} \phi(t)\,\frac{\sin (m + 1/2)t}{\sin t/2}\,dt$$
$$= \frac{1}{\pi}\int_0^{\pi} \phi(t)\,\frac{\sin mt}{t}\,dt + O(1),$$

while

$$S_m^*(0) \equiv \frac{1}{\pi}\int_0^{\pi} \phi(t)\,\frac{\sin mt}{t}\,dt$$
$$= \frac{1}{\pi}\sum_{n=3}^{\infty}\frac{2}{n\log n}\left\{\int_{n^{-1}-n^{-2}}^{n^{-1}-n^{-3}}\frac{\sin mt}{(t-n^{-1})}\,dt + \int_{n^{-1}+n^{-3}}^{n^{-1}+n^{-2}}\frac{\sin mt}{(t-n^{-1})}\,dt\right\}.$$

But

$$\sin mt = \sin (mn^{-1})\cos (m(t-n^{-1})) + \sin (m(t-n^{-1}))\cos (mn^{-1}),$$

so that

$$\pi S_m^*(0) = 4\sum_{n=3}^{\infty}(n\log n)^{-1}\cos (mn^{-1})\int_{n^{-3}}^{n^{-2}}\frac{\sin mt}{t}\,dt$$
$$= 4\left\{\sum_{n=3}^{[m^{1/3}]} + \sum_{n=[m^{1/3}]+1}^{[m^{1/2}]} + \sum_{n=[m^{1/2}]+1}^{\infty}\right\}$$
$$\equiv I_1 + I_2 + I_3.$$

Now, if $ma<1$, $a>b>0$,

$$\int_b^a \frac{\sin mt}{t}\,dt = \int_{mb}^{ma}\frac{\sin mt}{t}\,dt = O\{ma\},$$

and, if $mb>1$, $a>b>0$,

$$\int_b^a \frac{\sin mt}{t}\,dt = \int_{mb}^{ma}\frac{\sin t}{t}\,dt = O\left\{\frac{1}{mb}\right\} = O(1).$$

Hence

$$I_1 = O\left\{\sum_{n=3}^{[m^{1/3}]}(n\log n)^{-1}\frac{n^3}{m}\right\} = O\left\{\frac{1}{m}\sum_{n=3}^{[m^{1/3}]}\frac{n^2}{\log n}\right\} = o(1),$$

$$I_2 = O\left\{\sum_{n=[m^{1/3}]+1}^{[m^{1/2}]}(n\ \log\ n)^{-1}\right\}$$

$$= O\{\log\log m^{1/2} - \log\log m^{1/3}\} = O(1)$$

$$I_3 = O\left\{m\sum_{n=[m^{1/2}]+1}^{\infty}(n\log n)^{-1}n^{-2}\right\}$$

$$= o\left\{m\sum_{n=[m^{1/2}]+1}^{\infty}n^{-3}\right\} = o(1).$$

Therefore

$$S_m(0) = S_m^*(0) + O(1) = O(1).$$

We now apply Abel's partial summation to (1) and get

$$\tag{7} \begin{aligned} &\sum_{n=2}^{\infty}(S_n - S_{n-1})(\log n)^{-1/2} \\ &= S_1(\log 2)^{-1/2} + \sum_{n=2}^{\infty}S_n\{(\log n)^{-1/2} - [\log(n+1)]^{-1/2}\}, \end{aligned}$$

since $S_m(0) = O(1)$ and $(\log n)^{-1/2}\rightarrow 0$ as $n\rightarrow\infty$. But since $\log n$ is monotone,

$$\sum_{n=2}^{\infty}\{\,|\,(\log n)^{-1/2} - (\log[n+1]^{-1/2}|\,\} = (\log 2)^{-1/2},$$

and therefore (7) converges. This means that the point $x=0$ is contained in $E(Pl, f)$. But since we have already seen that the point $x=0$ is not contained in $E(L, f)$, this proves that

$$E(Pl, f) \not\subset E(L, f).$$

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