with the Hölder method of summability, where $i=1, 2, 3$. In the latter case, for example,

$$(x-1)S_{\infty,3}^{(k)} = 2S_{\infty,1}^{(k)} + \frac{1}{4}S_{\infty,0}^{(k)} + \frac{1}{3}$$
$$-\frac{1}{4}\left[(C, k) \text{ of } \sum_{1}^{\infty} \frac{1}{(2r-1)(2r+3)} X_r\right], \qquad (k > 5/2).$$

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TRIANGULATION OF THE MANIFOLD OF CLASS ONE*

BY S. S. CAIRNS

1. *Introduction.* In the present note, the writer shows that the triangulation method developed in an earlier paper† can be applied to divide a manifold of class one, as defined by Veblen and Whitehead,‡ into the cells of a complex. The manifold of class one includes the regular r-manifold of class C^n on a Riemannian space.§

2. *The Triangulation Theorem.* Let M_r be an arbitrary r-manifold of class one. A *coordinate system* is a correspondence between a point set, the *domain* of the system, on M_r, and a point set, called the *arithmetic domain*, in affine r-space. *Allowable coordinate systems* are a class of one-to-one correspondences whose properties are specified by axioms.||

Theorem. *If an r-manifold, M_r, of class one is covered by the domains of a finite set of allowable coordinate systems, it can be triangulated into the cells of a finite complex. Otherwise it can be triangulated into the cells of an infinite complex.*

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* Presented to the Society, December 28, 1934.

† *On the triangulation of regular loci*, Annals of Mathematics, vol. 35 (1934), pp. 579–587. Hereafter we refer to this paper as Triangulations.

‡ *A set of axioms for differential geometry*, Proceedings of the National Academy of Sciences, vol. 17 (1931), pp. 551–561; also, *The Foundations of Differential Geometry*, Cambridge Tract No. 29, 1932, Chapter 6, referred to below as Foundations.

§ Marston Morse, *The Calculus of Variations in the Large*, Colloquium Publications of this Society, vol. 18 (1934), Chapter 5.

|| Veblen and Whitehead, loc. cit.

We will discuss only the finite case. The other can then be treated by a ready extension.

3. *The Coordinates H.* LEMMA. *If M_r is covered by the domains of a finite set of allowable coordinate systems, then it can be covered by a finite set $(\gamma)\equiv(\gamma^1, \cdots, \gamma^n)$ of closed r-cells, where γ^i, $(i=1, \cdots, n)$, is on the domain of an allowable coordinate system H^i, and where the arithmetic domain of H^i is an r-simplex, a^i.*

For, let δ denote one of a finite set of domains covering M_r. Let δ' be the part of δ not covered by the other domains of the set. Since the domains are regions (Foundations, p. 78, Theorem 4), there exists, by the Heine-Borel theorem, a finite set of r-cells on δ covering δ'. Hence M_r can be covered by a finite set of r-cells $(\alpha)\equiv(\alpha^1, \cdots, \alpha^n)$, where (1) α^i, $(i=1, \cdots, n)$ is the domain of an allowable coordinate system H^i and (2) the arithmetic domain of H^i is a simplex (Foundations, p. 76, Theorem 2). It remains only to choose γ^i as a closed r-cell on α^i containing all points not covered by the other cells of the set (α).

(A) *We will denote by β^i, $(i=1, \cdots, n)$, an r-cell which contains γ^i and whose closure, $\overline{\beta}^i$, lies on α^i.*

An m-cell, $(m=0, \cdots, r)$, on α^i is called an H_m^i-cell if its image under H^i is a simplex. Now let σ_{m-1} be an $H_{m-1}^{i_1\cdots i_p}$-cell, assuming the term defined. If a point P can be joined to each point of σ_{m-1} by an H_1^j-cell, $(j\neq i_1, \cdots, i_{p-1})$, and if the totality of such cells is an m-cell, σ_m, then σ_m is called an $H_m^{i_1\cdots i_p j}$-cell or an *$H_m^{i_1\cdots i_p}$-cell* according as $j\neq i_p$ or $j=i_p$. A set of $(r+1)$ points will be called *$(j_1 \cdots j_q)$-dependent* if, using them in any order whatever, we can construct every conceivable $H_r^{i_1\cdots i_p}$-cell, where $(j_1 \cdots j_q)\supset(i_1 \cdots i_p)$.

(B) *The intersection $\alpha^j\cdot\alpha^k$ corresponds under H^j and H^k to subsets of a^j and a^k, respectively. The resultant of H^j and H^k is a homeomorphism*

$$(1) \qquad H^{jk}: \quad v_i = f_i(u), \qquad (i=1, \cdots, r),$$

between these subsets. By Axiom A_1, since the H's are allowable, H^{jk} is of class one (each f is continuous with its first derivatives) and regular (the jacobian of the f's is nowhere zero).*

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* Veblen and Whitehead, loc. cit.

The α's, β's, and γ's play the role of the cells similarly denoted in Triangulations, §3, and it is now possible to read §§4–7 of that paper, omitting §4, Lemma 1, and §4 (A), but making no other essential changes. We accordingly confine ourselves to a summary of the work, referring to Triangulations for details.

The vertices of an r-simplex constitute a *θ-set* if every angle* between a bounding $(r-1)$-simplex and an edge to the opposite vertex exceeds θ.

(C) *Let (P) be a set of $(r+1)$ points on M_r and let $(\beta^{j_1}, \cdots, \beta^{j_q})$ be the set of all the cells β [see* (A) *above] each containing a point of (P). Then, for any $\theta>0$, a distance $\rho(\theta)>0$ exists, so small that if (P) is a θ-set of diameter less than $\rho(\theta)$ in terms of some H^i-distance and H^i-angles, $(i=j_1, \cdots, j_q)$, then (P) is $(j_1 \cdots j_q)$-independent.* [Triangulations, §5(D) and §6, Theorem.]

4. *Construction of the Triangulation.* Let the r-simplex a^1, the arithmetic domain of H^1, be triangulated into simplexes, such that the vertices of each r-simplex constitute a θ_1-set of diameter less than $\rho(\theta_1)$, for some $\theta_1>0$. The cells which correspond under H^1 to these simplexes constitute a triangulation (σ^1) of α^1 into H^1-cells.

As the basis for a recurrent process we assume that we have, for some number j of the set $(1, \cdots, n-1)$, a triangulation (σ^j) of a neighborhood on M_r of $(\gamma^1+ \cdots +\gamma^j)$, where (1) the cells (σ^j) are $H^{i_1\cdots i_p}$-cells, $(i_1<i_2< \cdots <i_p\leqq j)$, and (2) the vertices of each r-cell of (σ^j) satisfy condition (C) in §3 for some angle of a set $(\theta_1, \cdots, \theta_j)$.

The $(j+1)$th step of the process modifies (σ^j) and extends it to cover a neighborhood of γ^{j+1}. Let P denote an interior vertex† of (σ^j) on $\bar{\beta}^{j+1}$. Let σ_m be any m-cell of the star of P, and let σ_{m-1} be the bounding $(m-1)$-cell of σ_m opposite P. By the hypotheses of the recurrency, σ_{m-1} is an $H_{m-1}^{i_1\cdots i_p}$-cell for some $(i_1, \cdots, i_p)$ and, applying §3(C), P and σ_{m-1} determine an $H_m^{i_1\cdots i_p j+1}$-cell σ_m'. Let σ_m' replace σ_m as σ_m denotes successively the various cells of the star of P. The totality of these replacements will be referred to as the *introduction of H^{j+1}-straightness*

* We define *distance* and *angle* by regarding the coordinates in arithmetic space as a rectangular cartesian system.

† A vertex of (σ^j) is called *exterior* if it is on the closure of $M_r-(\sigma^j)$. Otherwise it is called *interior.*

on the star of P. When H^{j+1}-straightness is thus introduced on the star of an interior vertex, a new complex is obtained covering the same points as (σ^j) [Triangulations, §7(C)].

Now let H^{j+1}-straightness be introduced successively on the stars of the other interior vertices of (σ^j) on $\bar{\beta}^{j+1}$. When this is done, let all cells with exterior vertices be dropped.* In the resulting complex,† $(\sigma^{j+1})^0$, every cell with all its vertices on $\bar{\beta}^{j+1}$ is an H^{j+1}-cell. Now consider a complex $(\sigma^{j+1})''$ made up of H^{j+1}-cells on β^{j+1}, such that (1) $(\sigma^{j+1})''$ covers the part of γ^{j+1} not on $(\sigma^{j+1})^0$ and (2) all overlapping cells of $(\sigma^{j+1})^0$ and $(\sigma^{j+1})''$ are H^{j+1}-cells.‡ By elementary subdivisions, let $(\sigma^{j+1})^0$ and $(\sigma^{j+1})''$ be subdivided until they have just a subcomplex in common. The set (σ^{j+1}) of all cells each in one of these subdivided complexes now satisfies all the requirements of the recurrent process, save perhaps the one involving §3(C). We can satisfy this one also by slightly displacing certain vertices in the elementary subdivisions above and imposing a new upper bound on the cells employed. This completes an outline of the $(j+1)$th step.

The nth step completes the triangulation of M_r into the cells of a simplicial complex.

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* Because the domains $(\alpha^i, \beta^i, \gamma^i)$ are nested [§3(A)], it is possible, by imposing an upper bound on the diameters of the cells employed, to ensure that the dropping of cells with exterior vertices will not uncover any points of a certain neighborhood of $(\gamma^1 + \cdots + \gamma^j)$. For details, see Triangulations, §7(A) and (B).

† See Triangulations, §7, Lemma 2, for a proof that $(\sigma^{j+1})^0$ is a complex. The present construction of $(\sigma^{j+1})^0$ is suggested on page 587 at the end of the proof.

‡ For further restrictions on $(\sigma^{j+1})''$ see Triangulations, under §7(D).