ON THE LIMIT OF A SEQUENCE OF POINT SETS

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A variable point P_n is said to approach the point P as its limit if to an arbitrary positive ϵ there corresponds an m such that

$$\overline{P_nP} < \epsilon, \qquad (n > m).$$

In other words, P is to have the property that every neighborhood of it contains almost all* the points P_n.

In attempting to generalize this definition to a sequence of point sets $M_1, M_2, \cdots$, one is naturally led to begin with a definition of the neighborhood of a set and then write down (Definition A_0) the last sentence of the last paragraph, replacing the letter P by M.

DEFINITION. *By the ϵ-neighborhood of a set M is meant the set of all points which have a distance $<\epsilon$ from some point of M. We shall denote it by $(\epsilon)_M$.*

DEFINITION A_0. *A point set M is called a limit of the sequence of sets $M_1, M_2, \cdots$, if every neighborhood of it contains almost all the sets M_i as partial sets.*

But the above definition is far from being useful, because the limit would then not be unique. In the first place, if the set M is a limit in the sense of Definition A_0, and if M has a cluster point C, then the set $M-C$ has also the property of being a limit of the sequence. Secondly every set containing M as a partial set is a fortiori a limit.

The first difficulty is overcome by requiring M to be closed, and the second difficulty is met by adding still another condition (γ):

DEFINITION A. *A set M is said to be the limit of the sequence of sets $M_1, M_2, \cdots$, if it has the following properties:*

(α) *M is closed.*

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* Thereby is meant that at most a finite number of the points P_i can lie outside the neighborhood.

(β) *For an arbitrary* $\epsilon > 0$, $(\epsilon)_M \supset M_i$* *for almost all indices* i.

(γ) *For an arbitrary* $\epsilon > 0$, $(\epsilon)_{M_i} \supset M$ *for almost all indices* i.

Observe that in the case of a sequence of points, (α) is fulfilled, (β) and (γ) become equivalent, and the definition reduces to the old one.

The following are immediate consequences of the definition:

(1) If a sequence of sets has a limit, the limit is unique.

(2) If a sequence $\mathfrak{S}$ of point sets has the limit M, every partial sequence of $\mathfrak{S}$ has the same limit M.

Further results hereby obtained consist of two fundamental criteria for the existence of a limit, when we restrict the sets of the sequence to lying in the same finite region of space. Given a sequence $\mathfrak{S}$ of sets $M_1, M_2, \cdots$, an L-point of $\mathfrak{S}$ shall be defined as a point which is the limit of a sequence of points $P_1, P_2, \cdots$, where each P_i belongs to the set M_i.

THEOREM A. *Let*

$$\mathfrak{S}: \qquad M_1, M_2, \cdots$$

be a sequence of point sets such that all the M_i*'s lie in the same finite region of space. Then* $\mathfrak{S}$ *has a limit when and only when, whatever partial sequence* $\mathfrak{S}_1$ *be selected from* $\mathfrak{S}$, *the set of L-points of* $\mathfrak{S}_1$ *coincides with the set of L-points of* $\mathfrak{S}$. *The limit of* M_i *is then the set of L-points of* $\mathfrak{S}$.

THEOREM B. *A necessary and sufficient condition for the sequence of sets* M_i, *lying in the same finite region of space, to have a limit is that, to an arbitrary positive* ϵ, *there corresponds an* M_m *such that*

(β') $(\epsilon)_{M_m} \supset M_i$ *for almost all indices* i,

(γ') $(\epsilon)_{M_i} \supset M_m$ *for almost all indices* i.

EXAMPLE 1. If each M_i is closed, M_1 is bounded, and $M_i \supset M_{i+1}$ for all i's, then a limit M exists and is equal to the set of points common to all the M_i's.

EXAMPLE 2. If $M_{i+1} \supset M_i$ and all the M_i's lie in the same finite region of space, then a limit M exists and is equal to the closed cover† of the set of points which belong to one of the M_i's.

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* Read: "the ϵ-neighborhood of M contains M_i as a partial set."

† The closed cover of a set is the sum of the set and its first derived set.

In the case where each M_i is a point, the meaning of Theorem A is obvious, while Theorem B leads directly to the fundamental criterion for the variable point P_n to approach a limit, namely, to an arbitrary $\epsilon > 0$ there corresponds an m such that $\overline{P_n P_{n'}} < \epsilon$, provided that $n, n' < m$.

But what do these theorems tell us when the set M_i corresponds to a point function?

To be exact, consider a sequence of functions $f_1(P), f_2(P), \cdots$, defined in the same bounded set N of $(n-1)$-dimensional space, and converging toward a limiting function $f(P)$, in each point P of N. Moreover, let the functions $f_i(P)$, $f(P)$ be bounded; that is, $|f_i(P)| < G$, $|f(P)| < G$, where G is the same number for all the functions. To each f_i corresponds then a bounded set M_i, formed of the points $(x_1, x_2, \cdots, x_{n-1}, x_n)$, where $P: (x_1, x_2, \cdots, x_{n-1})$ is a point of N and $x_n = f_i(P)$. Furthermore, all the sets M_i lie in the same finite region of space. Let M be the set corresponding to f as M_i to f_i. The following result is immediate. *If f_i is uniformly convergent, M_i has the closed cover of M as its limit.* But the converse is not true. For example, let N be the interval $(0, 1)$, and

$$f_i = \begin{cases} \epsilon_i & \text{when } 0 \leqq x \leqq \eta_i, \\ 1 - \epsilon_i & \text{when } \eta_i < x \leqq 1, \end{cases}$$

where $\epsilon_i > 0$, $\eta_i > 0$, $\epsilon_i > \epsilon_{i+1}$, $\eta_i < \eta_{i+1}$, $\lim_{n=\infty} \epsilon_n = 0$, $\lim_{n=\infty} \eta_n = 1/2$. Here f_i converges non-uniformly, while M_i has the closed cover of M as its limit.

Nevertheless it is true that *if N is closed and f is continuous, then f_i converges uniformly when M_i approaches M as its limit.* Thus under appropriate restrictions Theorem B is equivalent to the condition for uniform convergence, namely, f_i converges uniformly when and only when, to an arbitrary $\epsilon > 0$, there corresponds an m, independent of P, such that

$$|f_n(P) - f_{n'}(P)| < \epsilon, \qquad (m < n, n').$$

Under the same restrictions Theorem A may be translated as follows. *The function f_i is uniformly convergent when and only when, for every sequence of points $P_1, P_2, \cdots$ of N with the limit P, the sequence of numbers $f_1(P_1), f_2(P_2), \cdots$ has the limit $f(P)$.*

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