erally separable domains of diameter $1/n$ or less. By the above argument each G_n contains a countable subcollection G_n' covering space. Hence, space is completely separable and the theorem is established.

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A REDUCED SET OF POSTULATES FOR ABSTRACT HILBERT SPACE*

BY A. E. TAYLOR

1. *Introduction.* An abstract Hilbert space is a normed linear space, or *vector space*, of infinite dimensionality, with a norm based on a Hermitian *inner product*, defined for all pairs of elements in the space. The space is, moreover, separable and complete according to this norm. The usual postulate system for Hilbert space, which was first stated abstractly by J. von Neumann, consists of five groups of postulates, or nineteen in all.†

The purpose of the present paper is to demonstrate the redundancy‡ of a number of the postulates, and to present a system of eleven independent postulates for a normed linear space with a Hermitian inner product. The adjunction of three more postulates, each of which is independent of the first eleven and the remaining two, then gives us a system which is equivalent to that of von Neumann, that is, it defines an abstract Hilbert space, and it is categorical.

A special feature of this postulate system is that the abstract relation called equality, and denoted, as usual, by the symbol $=$, enters on an equal footing with the operations defined in the space.§ Three of the eleven postulates are concerned with this

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* Presented to the Society, December 1, 1934.

† J. von Neumann, *Mathematische Grundlagen der Quantenmechanik*, 1932. pp. 19–24; M. H. Stone, *Linear Transformations in Hilbert Space*, Colloquium Publications of this Society, vol. 15, pp. 2–4.

‡ Some of these redundancies were noted simultaneously by a fellow-student, Mr. Ivar Highberg, and myself.

§ The postulational treatment of equality in vector spaces was suggested by A. D. Michal in a critique of postulate systems. See this Bulletin, vol. 39 (1933), Abstract No. 339.

relation, giving its properties in connection with the operations. It is then demonstrated that the equality relation has the usual properties of an equivalence relation: reflexiveness, symmetry, and transitivity.

2. *A Normed, Linear Space.* Let K be a class of elements $x, y, \cdots$; and let C be the class of complex numbers $a, b, c, \cdots$. Let there be given in K a binary relation, called equality, and denoted by $=$, such that given an ordered pair of elements x, y from K, then either x bears the relation to y $(x=y)$, or it does not $(x\neq y)$. Let $+$ be the symbol of a binary operation, or rule of combination, defined throughout K. That is, to each ordered pair of elements x, y from K, there corresponds a unique entity, called their sum, and denoted by $x+y$. Let $\cdot$ be the symbol of a binary operation defined on the composite of C and K; that is, given $a\epsilon C$ and $x\epsilon K$, there is uniquely determined an entity, denoted by $a\cdot x$, or simply ax. Let $(\ ,\ )$ be the symbol of a binary operation, called the inner product, defined for all ordered pairs $x, y\epsilon K$, and yielding a unique entity, written (x, y).

The universe of discourse composed of the class K, the relation $=$, and the three operations $+, \cdot, (\ ,\ )$, is to be governed by the following eleven postulates, and it then forms a special kind of normed, linear space, as will be proved in §3.

1.0 *The class K contains at least one element.*

1.1 *If $x, y \epsilon K$, then $x+y \epsilon K$.*

1.2 *If $a \epsilon C$ and $x \epsilon K$, then $a\cdot x \epsilon K$.*

1.3 *If $x, y, (-1)\cdot y, x+(-1)\cdot y \epsilon K$, and if for each element $u \epsilon K$, $[x+(-1)y]+u=u$, then $x=y$.*

1.4 *If $x, y \epsilon K$, then $(x, y) \epsilon C$.*

1.5 *If $x, y, z, x+y \epsilon K$, and $(x+y, z), (x, z), (y, z) \epsilon C$, then $(x+y, z)=(x, z)+(y, z)$.*

1.6 *If $x, y \epsilon K$, and $(x, y), (y, x) \epsilon C$, then $(x, y)=\overline{(y, x)}$.*

1.7 *If $x, y, ax \epsilon K$, and $(ax, y), (x, y) \epsilon C$, then $(ax, y)=a(x, y)$.*

1.8 *If $x \epsilon K$ and $(x, x) \epsilon R$, then $(x, x)\geqq 0$, (R denotes the class of real numbers).*

1.9 *If $x \epsilon K$ and $(x, x)=0$, then $x+y=y$ for each y in K.*

1.10 *If $x, y \epsilon K$, and $x=y$, and if u is an arbitrary element in K such that $(x, u), (y, u) \epsilon C$, then $(x, u)=(y, u)$.*

This statement of the postulates is such as to make each one intelligible independently of the others. Thus, if 1.1, 1.2, and

1.4 hold, the remaining postulates may be greatly abbreviated. We also remark a double use of the symbols $=$, $+$, $\cdot$. No confusion will arise; the context suffices to make the meaning clear.

3. *General Theorems*. This section will be devoted to proving that the universe of discourse defined in §2 is a normed, linear space.* Questions of consistency and independence are discussed in §4.

DEFINITION. If $x \in K$, then $(-1)x \in K$, by 1.2, and we call this latter element $-x$ $(-x \equiv (-1)x)$. Similarly we define $x-y \equiv x+(-1)y$.

The following theorems are stated without proof.†

THEOREM 1. *If* $x, y \in K$, *and* $a \in C$, *then* $(x, ay) = \bar{a}(x, y)$.

THEOREM 2. *If* $x, y, z \in K$, *then* $(x, y+z) = (x, y) + (x, z)$.

THEOREM 3. *If* $x, y, u, v \in K$, *and* $a, b, c, d \in C$, *then*
$(ax + by, cu + dv) = a\bar{c}(x, u) + b\bar{c}(y, u) + a\bar{d}(x, v) + b\bar{d}(y, v)$.

DEFINITION. An element $z \in K$ such that $z+y=y$ for each $y \in K$, is called a zero element.

THEOREM 4. *There exists at least one zero element in* K. *If* x *is any element in* K, *then* $0 \cdot x$ *is a zero element, and* $x-x$ *is also.*

PROOF. By 1.0, there exists at least one element $x \in K$. Therefore $0 \cdot x \in K$, by 1.2. By 1.9, any element y such that $(y, y)=0$ is a zero element. But if x is arbitrary, $(0 \cdot x, 0 \cdot x)=0$, by 1.7. Also $(x-x, x-x)=0$, by Theorem 3. We shall presently prove that all zero elements are equal.

THEOREM 5. *If* $x, y \in K$, *then* $x+y=y+x$ (*Commutativity*). *If* $x, y, z \in K$, *then* $x+(y+z)=(x+y)+z$ (*Associativity*).

We shall prove the commutativity; the proof of associativity is quite similar, and will be left to the reader. We need merely show that $(x+y)-(y+x)$ is a zero element; the result then follows by postulate 1.3. Consider the inner product

$$((x + y) - (y + x), (x + y) - (y + x)).$$

* S. Banach, *Théorie des Operations Linéaires*, 1932, pp. 26–37 and p. 53. Banach calls them "espaces vectoriels normés."

† M. H. Stone, loc. cit., p. 4.

It is easily seen, on expanding this with the aid of Theorem 3, that the inner product is zero, and that therefore, by 1.9, $(x+y)-(y+x)$ is a zero element.

The next group of theorems establishes the familiar properties of the equality relation.

THEOREM 6. *If* $x, y \in K$, *and* $x=y$, *then* $(x, x)=(x, y)=(y, x)=(y, y)$.

The proof is accomplished by replacing the arbitrary element u of postulate 1.10, first by x, and then by y. We thus obtain $(x, x)=(y, x)$, and $(x, y)=(y, y)$. But since (x, x) is real, it follows by 1.6 that $(y, x)=(x, y)$. Thus the theorem is proved.

THEOREM 7. *If* $x, y, u, v \in K$, *and if* $x=y$ *and* $u=v$, *then* $x+u=y+v$.

PROOF. Consider the inner product

$$((x + u) - (y + v), (x + u) - (y + v)).$$

We find, upon expansion of this according to Theorem 3, that postulate 1.10 and Theorem 6 reduce the resulting expression to zero. Consequently the element $(x+u)-(y+v)$ is a zero element, and $x+u=y+v$. In quite similar fashion the following theorem is proved.

THEOREM 8. *If* $a, b \in C$, *and* $x, y \in K$, *and if* $a=b$ *and* $x=y$, *then* $ax=by$.

THEOREM 9. *The relation* $=$ *is an equivalence relation. That is, if* $x \in K$, *then* $x=x$ *(reflexiveness). If* $x, y \in K$, *and* $x=y$, *then* $y=x$ *(symmetry). If* $x, y, z \in K$, *and* $x=y$ *and* $y=z$, *then* $x=z$ *(transitivity).*

PROOF. We have already seen that $x-x$ is a zero element (Theorem 4). It follows at once that $x=x$. To prove the symmetry we consider the element $y-x$ and the inner product $(y-x, y-x)$. On expanding, we find the relation:

$$(y - x, y - x) = (y, y) - (y, x) - (x, y) + (x, x).$$

The expression on the right vanishes, by Theorem 6. Hence, by postulates 1.9 and 1.3, $y=x$. Finally, let $x=y$ and $y=z$. Then

$$(x - z, x - z) = (x, x) - (z, x) - (x, z) + (z, z).$$

From Theorem 6 and postulate 1.6 we have the relations:

$$(x, x) = (y, y) = (z, z), \quad (z, z) = (z, y) = (y, z),$$
$$(z, x) = (z, y) = (y, z), \quad (x, z) = (z, x) = (z, y).$$

From these it follows that $(x-z, x-z)=0$, so that $x=z$.

Postulates 1.0, 1.1 and Theorems 4, 5, 7, 9 suffice to show that the elements of K form an abelian group with respect to addition, using the equality relation.* The consequences of this are summed up in the following theorem.

THEOREM 10. *The class K forms an abelian group with respect to addition.*

The equation $x+y=z$ is uniquely (to within equal elements) solvable for x, and indeed $x=z-y$. In particular, the zero element in K is unique (that is, all zero elements are equal); and to each $x \in K$ corresponds a unique inverse, $-x$.

We shall denote the zero element in K by 0. This will occasion no ambiguity. The next group of theorems deals with the properties of multiplication by complex numbers.

THEOREM 11. *If $x \in K$, then $1\cdot x=x$. If $a \in C$, and 0 is the zero element in K, then $a\cdot 0=0$.*

PROOF. Consider the element $1\cdot x-x$; on expanding the inner product we find that $(1\cdot x-x,\ 1\cdot x-x)=0$. Therefore $1\cdot x=x$. Since 0 is the unique zero element, it may be written in the form $0\cdot x$, where x is arbitrary. It follows that $(0, 0)=0$. But $(a\cdot 0, a\cdot 0)=|a|^2(0, 0)=0$. Hence $a\cdot 0=0$.

THEOREM 12. *If $x, y \in K$ and $a, b \in C$, then $a(x+y)=ax+ay$, $(a+b)x=ax+bx$, $a(bx)=(ab)x$.*

PROOF. Each of the equalities is established by the method used in the proof of Theorem 5. For instance, by considering the element $a(bx)-(ab)x$, we show that its inner product with itself is zero, and hence that $a(bx)-(ab)x=0$.

This completes the demonstration that K is a linear space. In fact, the system of postulates $1.0-1.10$ is equivalent to the first two groups of postulates stated by von Neumann.†

* B. L. Van der Waerden, *Moderne Algebra*, 1930, vol. 1, p. 15.

† J. von Neumann, loc. cit., p. 19–21; von Neumann does not explicitly state any postulates for equality, but he uses all the properties which we have assumed or proved.

DEFINITION. By the *norm* of an element x we mean the non-negative number $(x, x)^{1/2}$; for convenience we shall write $\|x\| \equiv (x, x)^{1/2}$.

It is readily demonstrated that this norm has the usual properties. The proof is based on the Schwarz inequality; we summarize the results in the following theorem.*

THEOREM 13. *If $x, y \in K$, and $a \in C$, then*

$$|(x, y)| \leqq \|x\| \cdot \|y\|, \quad \|x\| \geqq 0, \quad \|0\| = 0,$$
$$\|x + y\| \leqq \|x\| + \|y\|, \quad \|ax\| = |a| \cdot \|x\|.$$

4. *Consistency and Independence of the Postulates.* Postulates $1.0-1.10$ are satisfied in any abstract Hilbert space, and are therefore consistent. As a specific example consider the class H_0 of infinite one-rowed matrices of complex numbers $x = (x_1, x_2, \cdots)$ such that the series $\sum_{i=1}^{\infty}|x_i|^2$ is convergent. If $y = (y_1, y_2, \cdots)$ is an element of H_0, we define $x = y$ if and only if $x_i = y_i$, $(i = 1, 2, 3, \cdots)$; and define

$$x + y = (x_1 + y_1, x_2 + y_2, \cdots),$$
$$a \cdot x = (ax_1, ax_2, \cdots), \quad (x, y) = \sum_{i=1}^{\infty} x_i \bar{y}_i.$$

This system forms a Hilbert space.† For the sake of uniformity, most of the independence examples have been formed from variations of this example. We present these examples below, with a brief explanation of each one; the details of verifying the postulates in each case are rather simple, and are, for the most part, left to the reader. The number of the example denotes the postulate which is *not* satisfied, thereby being proved independent.

EXAMPLE 1.0. Let K be any empty class. Then postulates $1.1-1.10$ are satisfied vacuously.

EXAMPLE 1.1. Let M be a positive real number, and consider the class of all infinite one-rowed matrices of complex numbers $x = (x_1, x_2, \cdots)$ in which at most M elements of the matrix are distinct from zero. This is a sub-class of the class H_0, and the

* J. von Neumann, loc. cit., pp. 21–22; M. H. Stone, loc. cit. pp. 4–5.

† M. H. Stone, loc. cit., pp. 14–15.

relation and operations are defined as in H_0. All the postulates are satisfied save that of closure under addition.

EXAMPLE 1.2. Consider the sub-class of H_0 in which all the elements of the matrices are *real* numbers, and let the relation and operations be defined as in H_0. All the postulates are satisfied save that of closure under multiplication by complex numbers.

EXAMPLE 1.3. Consider the class of complex-valued functions of the form

$$F + \sin nx, \qquad (-\infty < x < \infty),$$

where F is a complex number, and n is a non-negative integer. Then, if $f = F + \sin nx$, and $g = G + \sin mx$, we define

$$\begin{aligned} f &= g, \text{ if and only if } F = G \text{ and } n = m, \\ f + g &= F + G + \sin mx, \\ af &= aF + \sin nx, \\ (f, g) &= F \cdot \overline{G}. \end{aligned}$$

Each element in the class is determined by a complex number and a non-negative integer; the representation is unique. Postulates 1.0, 1.1, and 1.2 are clearly satisfied. Postulate 1.3 is not satisfied, for let $h = H + \sin px$ and suppose that $[f + (-1)g] + h = h$ for every such h. This implies the relation $F - G + H = H$. We infer that $F = G$, but we can say nothing about n and m. Postulates 1.4–1.8 are satisfied, as are 1.9 and 1.10 also.

EXAMPLE 1.4. Consider the class of infinite one-rowed matrices of complex numbers $x = (x_1, x_2, \cdots)$. No convergence condition is attached. Equality, addition, and multiplication are defined as usual. When x and y are elements of H_0, the inner product is defined as the series

$$(x, y) = \sum_{i=1}^{\infty} x_i \bar{y}_i.$$

For such elements the inner product is a complex number. Between all other pairs of elements it is defined as the infinite one-rowed matrix $(x_1\bar{y}_1, x_2\bar{y}_2, \cdots)$. Thus the inner product is not always a complex number. All the postulates save 1.4 are satisfied.

EXAMPLE 1.5. Consider the class H_0, with equality, addition, and multiplication defined as usual. Then, since the series $\sum_{i=1}^{\infty}|x_i|^2$ is convergent,

$$\lim_{n\to\infty} |x_n| = 0,$$

and for each matrix x there exists an integer i such that $|x_i| = \max \{|x_1|, |x_2|, \cdots\}$. Then if k is such that $|y_k| = \max \{|y_1|, |y_2|, \cdots\}$, we define $(x, y) = x_i \cdot \bar{y}_k$. This system satisfies all the postulates save 1.5, as is readily verified.

EXAMPLE 1.6. Consider the class H_0, with equality, addition, and multiplication defined as usual, and define the inner product

$$(x, y) = \max_{i=1, 2, \cdots} |y_i| \sum_{k=1}^{\infty} x_k \bar{y}_k.$$

Then clearly postulate 1.6, that of Hermitian symmetry, is not satisfied, whereas the other postulates *are* satisfied.

EXAMPLE 1.7. Consider the class H_0, with equality, addition, and the inner product defined as usual. For multiplication by complex numbers we define

$$a \cdot x = (R(a)x_1, R(a)x_2, \cdots),$$

where $R(a)$ is the real part of a. This system fails on postulate 1.7.

EXAMPLE 1.8. Consider the class H_0, with the definitions as in the Hilbert space of the consistency example, save for the inner product, which is the *negative* of the usual inner product: $(x, y) = -\sum_{i=1}^{\infty} x_i \bar{y}_i$. This leads to a negative-definite quadratic form:

$$(x, x) \leqq 0, \ (x, x) = 0 \text{ if and only if } x_i = 0, \qquad (i = 1, 2, \cdots).$$

Thus postulate 1.8 is independent.

EXAMPLE 1.9. Consider the class H_0, with equality, addition, and multiplication defined as usual, and the inner product defined to be zero for every pair of elements: $(x, y) \equiv 0$. This obviously satisfies all the postulates but 1.9.

EXAMPLE 1.10. Consider the class H_0, with addition, multiplication, and the inner product defined as usual. For equality we define $x = y$ if and only if

$$\sum_{i=1}^{\infty} |x_i|^2 = \sum_{i=1}^{\infty} |y_i|^2.$$

Then postulates 1.0–1.9 are satisfied, but 1.10 is not, as the following example shows. Let $x = (1, 2, 0, 0, \cdots)$ and $y = (2, 1, 0, 0, \cdots)$. Then $x = y$, for

$$\sum_{i=1}^{\infty} |x_i|^2 = \sum_{i=1}^{\infty} |y_i|^2 = 5.$$

But if $u = (1, 0, 0, \cdots)$ (u is obviously in H_0), we have $(x, u) = 1$, and $(y, u) = 2$.

These eleven examples prove the independence of the postulates.*

5. *Abstract Hilbert Space.* In this paragraph we shall consider the universe of discourse composed of the class K, its relation and operations, subject to the postulates $1.0 - 1.10$, and in addition, the following three.

2.1. *For each positive integer n, there exist elements $x_1, x_2, \cdots, x_n \epsilon K$ such that $a_1x_1 + \cdots + a_nx_n = 0$ if and only if $a_1 = \cdots = a_n = 0$.*

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* The postulates may easily be modified to yield an independent set for a *real* space, that is, a space closed under multiplication by real numbers (the class R). Obvious alterations only are required in §§3–4, and in most of the independence examples. Examples 1.2 and 1.7, however, will not serve, and may be replaced by the following.

EXAMPLE 1.2 (real). Consider the class of infinite one-rowed matrices $x = (x_1, x_2, \cdots)$, where $\sum_{i=1} x_i^2$ is convergent, and the x_i are real, rational numbers. Multiplication by a real number α is defined by $\alpha \cdot x = (\alpha x_1, \alpha x_2, \cdots)$, and the other definitions follow the usual model.

EXAMPLE 1.7 (real). Consider the class of infinite one-rowed matrices of real numbers $x = (x_1, x_2, \cdots)$ such that $\sum_{i=1}^{\infty} x_i^2$ is convergent. Let multiplication by a real number α be defined by $\alpha \cdot x = (N(\alpha)x_1, N(\alpha)x_2, \cdots)$, where $N(\alpha)$ is precisely α if α is an integer, and $N(\alpha)$ is the first integer smaller than α in all other cases.

2.2. *K is separable according to the norm* $(x, x)^{1/2} \equiv \|x\|$.
2.3. *K is complete** *(according to the same norm).*

DEFINITION.† When the system of K and its relation and operations satisfies postulates 1.0–1.10 and 2.1–2.3, it is called an abstract Hilbert space, and denoted by H.

It naturally occurs to one to inquire about the independence of the fourteen postulates, viewed as a single system. Before any steps can be taken in this direction, postulates 2.1–2.3 must be restated so as to be intelligible in case some of the other postulates are not satisfied. While this is possible, it leads to many difficulties of expression. I believe that the present treatment is more natural and convenient.

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* J. von Neumann, loc. cit., p. 24. These postulates are identical with the last three of von Neumann. The latter has proved (p. 37) that each of these postulates is independent of the remainder of his set.

† The reader may easily convince himself that this definition is justified, and that our postulate system is equivalent to that of von Neumann. This latter system is known to be categorical. That is, an isomorphism can be established between any two examples of abstract Hilbert space.