A NEW SOLUTION OF THE GAUSS PROBLEM ON $h(s^2d)/h(d)$*

BY GORDON PALL

The following demonstration of the well known formula

$$h(p^2d') = \sigma^{-1}\{p - (d' \mid p)\} h(d') \tag{1}$$

may be worth noting. Here $h(\Delta)$ denotes the number of classes of primitive integral binary quadratic forms of non-zero discriminant Δ; p is any prime $\geqq 2$; $\sigma = 1$ if $d' < -4$ or d' is a square, $\sigma = 2$ if $d' = -4$, $\sigma = 3$ if $d' = -3$; and if d' is positive but not square, σ is the least positive integer for which $p \mid u_\sigma$, (t_k, u_k) denoting the successive positive integral solutions of $t^2 - d'u^2 = 4$.

Let $r(n)$ denote the number of sets of representations of n by a representative system of primitive forms of discriminant $d = p^2d'$. If q is a prime such that $(d \mid q) = 1$,

$$r(p^2q) = 2\{p - (d' \mid p)\}. \tag{2}$$

For by II (5), (33), (23)–(24),†

$$r(p^2q) = r(p^2)r(q) = 2r(p^2) = 2\{1 + r'(p^2)\},$$

where $r'(p^2)$ equals the number $p - 1 - (d' \mid p)$ of solutions w of

$$(pw)^2 \equiv p^2d' \pmod{4p^2}, \quad \frac{w^2 - d'}{4} \text{ prime to } p, \quad (0 \leqq pw < 2p^2).$$

By Theorem 4 of I, extended to $d > 0$ in II, there is associated with each class (connoted by K, say) of primitive forms f of discriminant p^2d', a unique ambiguous class C, or two non-ambiguous classes C and C^{-1}, of primitive forms g of discriminant d'; C is characterized as representing any prime represented by K. By II (13), such forms satisfy, for all integers n,

$$f(p^2n) = \sigma g(n). \tag{3}$$

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† References are to the writer's two papers: I, Mathematische Zeitschrift, vol. 36 (1933), pp. 321–343; and II, Transactions of this Society, vol. 35 (1933), pp. 491–509.

Choose n to be a prime q represented by C and prime to d. Then $g(q)=2$ if C is ambiguous, $g(q)=1$ if $C\neq C^{-1}$. If a form f_1 is associated with a form g_1 not in C or C^{-1}, $f_1(p^2q)=\sigma g_1(q)=0$. Hence, by (2) and (3), p^2q is represented in exactly $\eta\{p-(d'|p)\}\sigma^{-1}$ classes K, where η is 1 or 2 according as q is represented in only one (ambiguous) or two (reciprocal) primitive classes of discriminant d'.

McGill University

ON A REDUCTION OF A MATRIX BY THE GROUP OF MATRICES COMMUTATIVE WITH A GIVEN MATRIX*

BY P. L. TRUMP

1. *Introduction.* Two $n\times n$ matrices A and B, with elements in any field F, are said to be similar in F if there exists a non-singular $n\times n$ matrix S, with elements in F, such that $S^{-1}AS=B$.

Ingraham† has given a method for finding the most general solution, with elements in F, of the matrix equation

$$P(X) = A,$$

where $P(X)$ is a polynomial with coefficients in F, and A is a square matrix with elements in F. A certain set of dissimilar solutions $X_1, X_2, \cdots, X_k$ were obtained in terms of which the complete system of solutions was seen to be in the form $S^{-1}X_iS$, where S is commutative with A. The X_i's are obviously commutative with A.

The purpose of this investigation is to determine the conditions under which two $n\times n$ matrices C and D are similar under transformations of the group $[S]$ of non-singular matrices S which are commutative with a certain $n\times n$ matrix A, where the matrices C and D are also commutative with A. We then seek to describe possible canonical forms to which such matrices

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* Presented to the Society, September 4, 1934. This paper with proofs and detail that are omitted here, is on file as a doctor's thesis at the Library of the University of Wisconsin.

† *On the rational solutions of the matrix equation* $P(X)=A$, Journal of Mathematics and Physics, vol. 13 (1934), pp. 46–50.