THEOREM 4. *If the boundary $\Gamma$ of the plane bounded connected and simply connected domain $\gamma$ contains an indecomposable continuum $D$, there is a prime end of $\gamma$ which contains $D$.*

Here, as in the development of Theorem 2, for each value of $j$ the set $\Gamma = \sum \Gamma_{ji}$. Consequently $\sum \Gamma_{ji} \supset D$. If for each of these $\overline{c(\Gamma_{ji}) \cdot D} \supset D$, then the set $\sum \Gamma_{ji} \cdot D$ is nowhere dense in $D$ and $[\Gamma_{ji}]$ does not cover $D$. But as none of $[\Gamma_{ji}]$ can have $\overline{c(\Gamma_{ji}) \cdot D} \not\supset D$ unless $D \cdot c(\Gamma_{ji}) = 0$, in view of Lemma 4, there must for every value of $j$ be one of $[\Gamma_{ji}]$ which contains $D$. The proof now follows lines almost identical with those of Theorem 2.

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# PROJECTIVE DIFFERENTIAL GEOMETRY OF CURVES

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In a fundamental paper* on the projective differential geometry of curves, L. Berzolari obtained canonical expansions representing a curve $C$ immersed in a linear space $S_n$ in a neighborhood of one of its points $P_0$. The vertices of the coordinate simplex yielding Berzolari's canonical form are covariantly related to the curve, while the unit point may be any point of the rational normal curve $\Gamma$ which osculates $C$ at $P_0$. It is the purpose of the present paper to define a covariant point on $\Gamma$ which can be chosen as a unit point so as to produce final canonicalization of the power series expansions of Berzolari.

It will be observed that the usual methods of defining a point on $\Gamma$ for the cases $n=2$ and $n=3$ depend on configurations† that do not possess suitable analogs in $n$-space. Hence it appeared for some time that the problem called for different procedures in spaces of different dimensionality. Special devices

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* L. Berzolari, *Sugli invarianti differenziali proiettivi delle curve di un iperspazio*, Annali di Matematica, (2), vol. 26 (1897), pp. 1–58.

† E. P. Lane, *Projective Differential Geometry of Curves and Surfaces*, pp. 12–27.

were found by S. B. Murray and the author* for the spaces $S_4$ and $S_5$; however, like the methods used in the plane and in $S_3$ these seem not to admit generalization. It is to be shown here that, with the help of a suitably chosen linear complex, the general problem for $n>3$ may be solved.

Local power series expansions representing an analytic curve $C$ immersed in a linear space $S_n$ of $n$ dimensions ($n>3$) in a neighborhood of an ordinary point $P_0$ may be written† in the form,

$$(1)\qquad \begin{aligned} x_0 &= 1, \\ x_i &= x_1^i + a_i x_1^{n+3} + b_i x_1^{n+4} + \cdots, \qquad (i = 2, \cdots, n), \end{aligned}$$

wherein $x_0, \cdots, x_n$ are homogeneous projective point coordinates, and the coefficients $a_i$, $b_i$, etc. are complex numbers, $a_{n-1}$ being zero and $a_n$ different from zero. The equations of the osculating rational normal curve $\Gamma$ of $C$ at $P_0$ are

$$x_i = x_1^i, \qquad (i = 0, \cdots, n).$$

The vertices of the coordinate simplex will be denoted by $P_0, \cdots, P_n$, where $P_i$ is the point for which

$$x_i = 1, \qquad x_j = 0, \qquad (j = 0, \cdots, n; j \neq i).$$

The point $P_n$ is the intersection that is distinct from $P_0$ of the curve $\Gamma$ and the principal hyperplane‡ of $C$ and $\Gamma$; the vertex $P_i$, $(i=1, \cdots, n-1)$, is the intersection of the osculating space $S_{n-i}$ of $\Gamma$ at $P_n$ and the osculating space $S_i$ of $C$ at $P_0$. The unit point $U(1, \cdots, 1)$ is any point on $\Gamma$ distinct from the points $P_0$ and $P_n$.

Homogeneous line coordinates $p_{ij}$ of the line joining points $X(x_0, \cdots, x_n)$ and $Y(y_0, \cdots, y_n)$ will be defined by

* See Murray, *Curves in Four-Dimensional Space*, Chicago master's dissertation, 1934, and Wilcox, *Curves in Five-Dimensional Space*, Chicago master's dissertation, 1933.

† Berzolari, loc. cit., p. 2. We shall say that $P_0$ is an ordinary point of $C$ in case (1) $C$ is not hyperosculated at $P_0$ by any of its linear osculants or by its osculating rational normal curve $\Gamma$, and (2) $C$ and $\Gamma$ have at $P_0$ a principal plane not contained in their osculating hyperplane at $P_0$. For the definition of principal plane see Berzolari, loc. cit., p. 18.

‡ Berzolari, loc. cit., p. 19.

$$p_{ij} = x_i y_j - x_j y_i, \qquad (i, j = 0, \cdots, n;\ i < j).$$

The coordinates of the line $l_{hk}$ joining $P_h$ and $P_k$ $(h<k)$ are given by

$$p_{ij} = \begin{cases} 1, & \text{when } i = h \text{ and } j = k, \\ 0, & \text{when } i \neq h \text{ or } j \neq k. \end{cases}$$

In the totality of linear complexes in the ambient space $S_n$ there is a two-parameter family containing all lines $l_{hk}$ except $l_{0,n}$, $l_{1,n-1}$, and $l_{3,n}$. The equation of this family is

$$\lambda p_{0,n} + \mu p_{1,n-1} + \nu p_{3,n} = 0, \tag{2}$$

wherein $\lambda$, $\mu$, $\nu$ are homogeneous parameters. In the family (2) there is a unique complex having $(n+3)$-line contact with the tangent developable of the curve $C$ at the line $l_{0,1}$. With the help of expansions (1) its equation is found to be

$$\begin{aligned}(n - 2)(n - 3)p_{0,n} - n(n - 3)p_{1,n-1} \\ - (n - 2)(n + 3)a_n p_{3,n} = 0.\end{aligned} \tag{3}$$

The locus of all lines of the complex (3) through the point $P_n$ is a hyperplane $\pi$ whose equation is

$$(n - 3)x_0 - (n + 3)a_n x_3 = 0.$$

If we demand that the unit point $U$ shall lie in this hyperplane, we have

$$a_n = \frac{n - 3}{n + 3};$$

hence we obtain the following result.

*An analytic curve $C$ immersed in a linear space of $n$ dimensions may be represented in a neighborhood of one of its ordinary points $P_0$ by local power series expansions of the form* (1), *in which $a_n = (n-3)/(n+3)$. For this canonical form the unit point is one of the intersections distinct from $P_n$ of the hyperplane $\pi$ with the osculating rational normal curve of $C$ at $P_0$.*

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