these questions, will be awaited with great interest. It may be that still other points of view will be necessary to complete the foundations of mathematics so well begun by the Hilbert proof-theory.

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SZEGÖ ON JACOBI POLYNOMIALS

Asymptotische Entwicklungen der Jacobischen Polynome. By Gabriel Szegö. Schriften der Königsberger Gelehrten Gesellschaft. Jahr 10, Naturwissenschaftliche Klasse, Heft 3, 1933, pp. 35–111 (1–77).

One of the interesting features in the development of analysis in the twentieth century is the remarkable growth, in various directions, of the theory of orthogonal functions. Two brilliant achievements on the threshold of this century—Fejér's paper on Fourier series and Fredholm's papers on integral equations—have been acting as a powerful inspiring source of attraction, inviting analysts to delve deeper into the theory of orthogonal functions and their applications. First come, due to their simplicity, the trigonometric functions $\{\sin mx, \cos mx\}$ which serve as a yardstick for orthogonal functions in general. Next we may consider orthogonal polynomials, of which Jacobi polynomials are a special case.

Let us recall the general definition of orthogonal polynomials. A weight-function $p(x)$, non-negative in a given interval (a, b), finite or infinite, and such that all "moments" $\int_a^b p(x)x^r dx = \alpha_r$ exist, $(r=0, 1, 2, \cdots)$, with $\alpha_0>0$, gives rise to a unique system of orthogonal and normal polynomials $\phi_n(x) = a_n x^n + \cdots$, $(n=0, 1, \cdots; a_n>0)$, so that

$$\int_a^b p(x)\phi_m(x)\phi_n(x)dx = 0, \ (m \neq n), \qquad = 1, \ (m = n), \quad (m, n = 0, 1, \cdots). \tag{1}$$

On the basis of (1), we obtain the following expansion of an "arbitrary" function:

$$f(x) \sim \sum_{n=0}^{\infty} f_n\phi_n(x), \quad \text{with} \quad f_n = \int_a^b p(x)f(x)\phi_n(x)dx, \tag{2}$$

and this constitutes the most interesting and important application of the polynomials $\phi_n(x)$ in analysis, as well as in mathematical physics, mathematical statistics, etc.

The oldest and best known are Legendre polynomials, derived from (1) with (a, b) finite, say $(-1, 1)$, and $p(x) \equiv 1$. Their direct generalization are Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$: $(a,b) = (-1,1)$, $p(x) = (1-x)^\alpha(1+x)^\beta$, $\alpha,\beta > -1$. In case of an infinite interval, the most important are the polynomials of Laguerre: $(a, b) = (0, \infty)$, $p(x) = x^\alpha e^{-x}$, $\alpha > -1$, and those of Hermite: $(a, b) = (-\infty, \infty)$, $p(x) = e^{-x^2}$. These four kinds of orthogonal polynomials constitute what may be considered as one single family of "classical" polynomials, where Jacobi polynomials, from many points of view, represent the most typical member. In fact, by assigning to α, β certain finite or limiting values, we get Legendre polynomials $(\alpha=\beta=0)$, trigonometric polynomials $(\alpha=\beta=-1/2)$, also the polynomials of Laguerre and Hermite. A study of Jacobi polynomials is therefore of special value and interest.

In the book under review Professor Szegö, whom we are now happy to greet in our midst, takes up the *asymptotic properties of* $P_n^{(\alpha,\beta)}(x)$, that is, their behavior when the degree n increases indefinitely. A glance at the expansion (2) shows at once the importance of these properties, for they influence to the greatest extent the convergence (or summability) of (2); they also play a predominant part in many other problems, like mechanical quadratures. To this study Professor Szegö brings his well known mastery of the finest tools of analysis, and his lucidity and elegance of exposition, which are worthy acquisitions from his eminent teacher Fejér.

The asymptotic expression for $P_n^{(\alpha,\beta)}(x)$, with arbitrary α, $\beta(>-1)$, was derived by Darboux, that for Legendre polynomials by Laplace. Trigonometric functions are exclusively employed in these expressions. While this makes for simplicity, we also find great inconvenience, due to the fact that the expressions in question hold *inside* the interval of orthogonality only. Here Szegö utilizes *Bessel functions*, instead of trigonometric ones,* and in this way he is able to derive asymptotic expressions *uniformly valid in any left-hand neighborhood of the end point* $x=1$. This is the main object of the book, the importance of which is evident to any worker in this field. It suffices to recall the laborious considerations and computations necessitated in the past by the lack of such asymptotic expressions, when evaluating for Legendre polynomials, for example, Lebesgue constants corresponding to the end points $x=\pm 1$. Szegö himself shows the application of his asymptotic expressions to various interesting problems involving Jacobi polynomials. Their solution now becomes strikingly simple.

The book consists of four chapters preceded by an Introduction, where one finds an outline of the general purpose of this work and of the methods employed, and is closed by an Appendix dealing with the so-called "Associated Functions". The methods employed and the results obtained are an extension of those to be found in a previous paper by the author on Legendre polynomials.

The first chapter is devoted to marshalling necessary data and formulas, and to formulating the fundamental results obtained. We first find here the definition, through the generating function, of $P_n^{(\alpha,\beta)}(x)$ for any real α, β (the book deals chiefly with the *orthogonal* Jacobi polynomials, that is, α, $\beta>-1$), with emphasis upon the symmetric case of "ultraspherical" polynomials $P_n^{(\mu)}(x)$ ($=$const. $xP_n^{(\alpha,\alpha)}(x)$, $\mu=\alpha+1/2$). This definition yields at once the explicit expression of $P_n^{(\alpha,\beta)}(x)$ and some of their fundamental properties:

$$P_n^{(\alpha,\beta)}(x) \equiv (-1)^n P_n^{(\beta,\alpha)}(-x); \quad \frac{d}{dx}P_n^{(\alpha,\beta)}(x) = \frac{n+\alpha+\beta+1}{2}P_{n-1}^{(\alpha+1,\beta+1)}(x).$$

The first property reduces the discussion of $P_n^{(\alpha,\beta)}(x)$ in $(-1, 1)$, which is the interval of orthogonality, if α, $\beta>-1$, to that in $(0, 1)$; the second property is a very important one; it shows, for example, that the asymptotic expression for $P_n^{(\alpha,\beta)}(x)$ can be differentiated and thus throws more light on the behavior

* For Legendre polynomials, this was partly done previously by Watson and Hilb.

of the remainder. The author next gives some properties of Bessel functions $J_\nu(\xi)$ of the first kind, emphasizing, for future needs, their asymptotic behavior for $\xi \to +0$ and $\xi \to +\infty$, also max $\xi^{1/2}|J_\nu(\xi)|$ for $\xi \geqq 0$. He further introduces,as a generalization of $J_\nu(\xi)$, certain functions $J_{\nu,m}(\xi)$ and, by expressing these in terms of $J_\nu(\xi)$, he shows how to estimate their order of magnitude for $\xi \to 0$, $\xi \to \infty$. The chapter closes by stating the fundamental theorems of this book giving the asymptotic expressions of $P_n^{(\mu)}(\cos\theta)$ (Theorem 1) and of $P_n^{(\alpha,\beta)}(\cos\theta)$ (Theorem 2). Thus

$$(3)\quad P_n^{(\mu)}(\cos\theta) \sim \sum_{m=0}^{\infty} f_m(\theta) \frac{\theta^{m-\mu+1/2}}{(n+\mu)^{-m\mu+1/2}} J_{\mu-m-1/2}[(n+\mu)\theta], \qquad (0 < \theta < \pi).$$

Here the essentially novel and important feature, as pointed out above, *is the use of Bessel functions.* The $f_m(\theta)$ are "elementary" functions, regular for $0 \leqq \theta < \pi$; the first few are given explicitly. The author also gives, for $n \to \infty$, an o-estimate of the remainder in the above expressions, broken up after a certain number of terms, for $\theta \geqq c/n$ and $\theta \leqq c/n$, (c fixed, >0). We also find here an asymptotic expansion of $P_n^{(\mu)}(\cos\theta)$ in terms of the orthodox trigonometric functions. The wider power of the asymptotic expansions employing Bessel functions is illustrated by the fact that the principal term alone suffices to improve Darboux's classical formula for $P_n^{(\alpha,\beta)}(x)$. Various possible applications of the foregoing expansions are indicated, one of the most important of which deals with mechanical quadratures (Theorem 3, see below).

Chapter 2 is devoted to the proof of Theorem 1, the asymptotic expansions of $P_n^{(\mu)}(\cos\theta)$. The main point is the expression of $P_n^{(\mu)}(x)$ as a contour integral (making use of the generating function). By choosing properly the contour of integration, so that the integrals involved become identical with those figuring in the definition of $J_\nu(\xi)$, we obtain (3). Another choice of the contour yields the orthodox (trigonometric) expansion for $P_n^{(\mu)}(\cos\theta)$.

Chapter 3 in its first half deals with Theorem 2, the asymptotic expansion of $P_n^{(\alpha,\beta)}(\cos\theta)$ (for any real α, β; $0 < \theta \leqq \pi - \epsilon$; $\epsilon > 0$, arbitrarily fixed, $0 < \epsilon < 1$). The analysis is again based on the integral representation of $P_n^{(\alpha,\beta)}(x)$ and requires here a more penetrating investigation of the generating function and the introduction of the generalized functions $J_{\nu,m}(\xi)$. The author shows that the remainder for $\theta \leqq c/n$ can be estimated in another way, by writing

$$n^{-\alpha} P_n^{(\alpha,\beta)}\left(\cos\frac{u}{n}\right) \sim \phi_0(u) + \frac{\phi_1(u)}{n} + \frac{\phi_2(u)}{n^2} + \cdots,$$

(where $\phi_m(u)$ are integral functions of u), and making use of the estimate already arrived at for $\theta \geqq c/n$. He further points out that for integral values of α there exists a certain interval for θ where the above expansion converges in the ordinary sense (a similar remark applies to the expansion (3)). Finally, a more precise o-estimate of the remainder is obtained, for some special values of α, by o-estimating the mth term of the expansion in question. The second half of Chapter 3 gives various interesting applications of the foregoing formulas. We thus find, for example,

$$(4)\qquad \begin{aligned} |P_n^{(\alpha,\beta)}(x)| &\leqq A_n^{-1/2}(1-x)^{-\alpha/2-1/4}, && (\alpha \geqq -1/2), \\ |P_n^{(\alpha,\beta)}(x)| &\leqq A_n^{\max(\alpha,-1/2)}, && -1+\epsilon \leqq x \leqq 1), \end{aligned}$$

where A is an appropriate constant, independent of n. We are also enabled to treat in a very simple manner the asymptotic behavior $(n\rightarrow\infty)$ of

$$\max\,(1-x)^{\lambda}(1+x)^{\mu}\left|\,P_n^{(\alpha,\beta)}(x)\right|,\quad(-1\leqq x\leqq 1),\ (\lambda,\mu\text{ fixed},\ \geqq 0),\tag{5}$$

which, for $\lambda=\alpha/2+1/4$, $\mu=\beta/2+1/4$, was treated by S. Bernstein by means of the differential equation for Jacobi polynomials. With $\alpha=\beta=0$, $\lambda=\mu=1/4$ or $\alpha=\beta=1$, $\lambda=\mu=3/4$ we obtain, respectively, the boundedness in $(-1,1)$ of $n^{1/2}(1-x^2)^{1/4}|P_n(x)|$ (Stieltjes), $n^{-1/2}(1-x^2)^{3/4}|P_n'(x)|$ (Kogbetliantz) ($P_n=$Legendre polynomial). Such estimates as (4) or (5) are very important when we are discussing, for example, the convergence of (2). In a similar manner, we can readily estimate asymptotically $(n\rightarrow\infty)$ many definite integrals involving $P_n^{(\alpha,\beta)}(x)$, such as

$$\int_{-1}^{1}\left|\frac{d}{dx}P_n^{(\alpha,\beta)}(x)\right|dx,\quad\int_{-1}^{1}(1-x)^{\lambda}(1+x)^{\mu}\left|\,P_n^{(\alpha,\beta)}(x)\right|,\quad(\lambda,\mu\text{ fixed},\ >-1).$$

Finally comes the most important application, the asymptotic estimate of the so-called Lebesgue constants

$$\rho_n^{(\alpha,\beta)}(x)=\int_{-1}^{1}(1-t)^{\alpha}(1+t)^{\beta}\left|\sum_{m=0}^{n}\phi_m(x)\phi_m(t)\right|dt,\tag{6}$$

where $\phi_m(x)$ are *normalized* Jacobi polynomials, *at the end point* $x=1$. These constants, as is well known, are of paramount importance in the discussion of the convergence properties of (2). The estimate of $P_n^{(\alpha,\beta)}(1)$ now follows very readily from the above asymptotic expressions, due to the fact that they hold uniformly in any left-hand neighborhood of $x=1$ (in contrast with the classical ones). The climax is reached in the closing section where the author proves the "*equiconvergence theorem*". Let the L-measurable function $f(x)$ be such that

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}|\,f(x)|\,dx,\ \int_{-1}^{1}(1-x)^{\alpha/2-1/4}(1+x)^{\beta/2-1/4}|\,f(x)|\,dx,\ (\alpha,\beta>-1),$$

exist. Denote by $s_n(x)$ the nth partial sum of the expansion (2) of $f(x)$ in series of Jacobi polynomials, by $\sigma_n(x)\equiv\sigma_n(\cos\theta)$, the nth partial sum of the Fourier cosine series expansion of $F(\theta)=(1-\cos\theta)^{\alpha}(1+\cos\theta)^{\beta}f(\cos\theta)|\sin\theta|$. Then

$$\lim_{n\rightarrow\infty}\left[s_n(x)-(1-x)^{-\alpha-1/2}(1+x)^{-\beta-1/2}\sigma_n(x)\right]=0,\qquad(-1<x<1),$$

uniformly for $-1+\epsilon\leqq x\leqq 1-\epsilon$. Moreover, if $f(x)$ is bounded in $(-1,1)$, then we can compare (2) with the ordinary Fourier series expansion of the function $f(\cos\theta)$ itself. *Thus the discussion of the convergence* (*or summability*) *of* (2) *is reduced to that of an ordinary Fourier series*, which is the goal (of course, not always attainable) for all expansions in series of orthogonal functions.

Chapter 4 is devoted to the Newton-Cotes mechanical quadrature formula based on the Lagrange interpolation formula *with the zeros of a Jacobi polynomial as abscissas*:

$$\int_{-1}^{1}f(x)dx=\sum_{i=0}^{n}\lambda_i f(x_i)+R_n\qquad\lambda_i=\int_{-1}^{1}\frac{\phi_n(x)}{(x-x_i)\phi_n'(x_i)}dx,\tag{7}$$

$$[\phi_n(x)=\text{normalized }P_n^{(\alpha,\beta)}(x),\ P_n^{(\alpha,\beta)}(x_i)=0].$$

The novelty and interest of the formula (7) lies in the fact that *the weight function* $p(x) = (1-x)^\alpha\ (1+x)^\beta$ *is here omitted* (in contrast with the classical formulas of mechanical quadratures of Gauss' type). On the basis of recent investigations of Fejér and Pólya, the question as to the general convergence of (7), that is, as to the existence of $\lim_{n\to\infty} R_n = 0$ for all R-integrable functions $f(x)$, essentially depends upon the signs of the "Cotes coefficients" λ_i. The author shows here that

$$\text{(8)} \qquad \operatorname{sgn} \lambda_i = \operatorname{sgn} K_n(x_i), \quad K_n(x) = \sum_{m=0}^{n} \phi_m(x) \int_{-1}^{1} \phi_m(t)dt.$$

Making use of the Darboux formula for $\sum_{m=0}^{n} \phi_m(x)\phi_m(t)$ and of some orthogonality properties of $K_n(x)$, Szegö is able to investigate the signs of $\{\lambda_i\}$ for some special α, β. A similar investigation for any α, β (>-1) is achieved through the improved (see above) Darboux asymptotic expression for $P_n^{(\alpha,\beta)}(\cos\theta)$. We thus conclude (Theorem 3, Chapter 1): (a) if max $(\alpha, \beta) > 3/2$, then there exists a function $f(x)$, continuous in $(-1, 1)$, for which (7) diverges; (b) if max $(\alpha, \beta) < 3/2$, then (7) converges for all $f(x)$ R-integrable in $(-1, 1)$. The case (b) also holds for $\alpha = \beta = 3/2$.* Special cases of Theorem 3 are due to Fejér. The chapter closes with the case $-1/2 \leqq \alpha = \beta \leqq 0$, where it is shown that all λ_i are positive.

The Appendix deals briefly with the application of the methods of Chapter 2 in order to obtain asymptotic expansions of the *associated functions*

$$P_n^\nu(\cos\theta) = e^{i\nu\pi}\cdot\frac{\Gamma(\nu+1/2)}{\Gamma(1/2)}(2\sin\theta)^\nu\cdot\frac{1}{2\pi i}\int_C (1 - 2w\cos\theta + w^2)^{-\nu-1/2} w^{n+\nu} dw.$$

Following traditional lines, we must try to find some fault with the book under review. We did find one fault—to be sure, of measure zero. The author in setting up the asymptotic expansions for $P_n^{(\mu)}(\cos\theta)$, $P_n^{(\alpha,\beta)}(\cos\theta)$, uses what he calls "elementary functions" $(f_m(\theta), f_{m,\nu}(\theta))$. This designation seems to the reviewer rather vague and thus unnecessary.

At the end of Chapter 1 the author promises to give, by similar methods, an investigation of the asymptotic nature of the polynomials of Laguerre and Hermite. The realization of this promise will be eagerly awaited, we feel certain, by all those who are interested in analysis in general, and in particular in the vast fruitful field of orthogonal polynomials.

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* Probably also for $\max(\alpha, \beta) = 3/2$, $\alpha \neq \beta$ (Szegö).