A NOTE ON UNITS IN SUPER-CYCLIC FIELDS

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1. *Comparison of Two Known Results Concerning Cyclotomic Units.* Kummer* first showed that if

$$\zeta = e^{2i\pi/l}$$

with l an odd prime, and if η is a unit in $k(\zeta)$ such that

$$\eta \equiv a \pmod{l},$$

where a is a rational integer, then

$$\eta = \rho^l,$$

where ρ is in $k(\zeta)$, provided none of the Bernoulli numbers

(1) $$B_1, B_2, \cdots, B_d, \qquad (d = (l-3)/2),$$

is divisible by l. Kummer's proof of this depended on the fact that under the assumptions mentioned there exists an integer c prime to l such that

(2) $$\eta^c = E_1^{a_1} E_2^{a_2} \cdots E_d^{a_d}.$$

Here

$$E_n = \prod_{i=0}^{d} \epsilon(\zeta^{r^i})^{r^{-2in}},$$

$$\epsilon = \left(\frac{(1-\zeta^r)(1-\zeta^{-r})}{(1-\zeta)\,(1-\zeta^{-1})}\right)^{1/2}.$$

From this we obtain an identity in an indeterminate x by adding a certain multiple of

$$\frac{x^l - 1}{x - 1}.$$

Setting $x = e^v$, taking logarithms and differentiating $2n$ times, $(n = 1, 2, \cdots, d)$, we find, using relations in another paper,†

* Journal für Mathematik, vol. 40 (1850), p. 128.

† Transactions of this Society, vol. 31 (1929), pp. 619–620, relations (4) and (5).

$$a_1 \equiv a_2 \cdots \equiv a_d \equiv 0 \pmod{l},$$

which is the result.

By using a quite different method, Hilbert* gave proof that if

$$\eta \equiv 1 \pmod{\lambda^l}, \qquad \lambda = 1 - \zeta$$

and $k(\zeta)$ is a regular field, then $\eta = \rho^l$.

A field $k(\zeta)$ is said to be regular if and only if l is prime to its class number. It is known that this condition is equivalent to the statement that the set (1) contains no numbers divisible by l.

Comparing the different forms of η in the two statements of Kummer and Hilbert, we note that if $\eta = a + \theta l$, where θ is in $k(\zeta)$, we may write $\theta = b + \lambda\theta_1$ where b is rational, and obtain $\eta = a + lb + \lambda^l\omega$. Now $(a + lb)$ is not necessarily equal to 1, so the two forms are not the same.

Hilbert's proof of the result as stated by him depended on his theory of class-fields. It was reproduced by Landau† who commented on the great length of the proof and the complexity of one of the lemmas involved, that is, the existence of a system of relative fundamental units in a Kummer field.

In the present paper I shall consider further the principles involved in the demonstration of this theorem and give an extension of it involving super-cyclic fields. I shall also consider analogous questions in connection with the cyclotomic field which is not regular. The proofs, in the main, will be merely sketched.

2. *A Theorem Concerning Primary Units in Super-Cyclic Fields.* Furtwängler‡ gave the result that if K contains the field $k(\zeta)$ and if the class number of K be $H = l^h q$, $q \not\equiv 0 \pmod{l}$, and a basis for the Abelian group formed by the qth powers of the ideal classes of K be $C_1, C_2, \cdots, C_e$, then there exists a basis for the singular primary numbers in K, $\omega_1, \omega_2, \cdots, \omega_e$, such that any singular primary number in K may be written in the form $\omega_1^{a_1}\omega_2^{a_2} \cdots \omega_e^{a_e}\alpha^l$. Also, corresponding to any singular primary number belonging to the basis, there is an ideal class C belonging to the basis of the so-called irregular class group.

* Werke, vol. 1, p. 287.

† *Vorlesungen über Zahlentheorie*, vol. 3, p. 240, p. 258.

‡ Mathematische Annalen, vol. 43 (1907), p. 18.

The above shows that if we have the primary units in K, that is, a unit η such that

$$\eta \equiv \gamma^l \pmod{\lambda^l},$$

it follows that if the field K has a class number which is prime to l, then no C exists and therefore no singular primary number. Hence η is an lth power in K.

The above argument can be put in somewhat different form by employing the law of reciprocity

$$\left\{\frac{\alpha}{\beta}\right\} = \left\{\frac{\beta}{\alpha}\right\},$$

where each member denotes an lth power character in K and α is a primary integer in K. As a special case of this we have*

$$\left\{\frac{\omega}{\beta}\right\} = 1,$$

where ω is a singular primary number in K. Let $\beta = \mathfrak{p}^h$, where $\mathfrak{p}$ is a prime ideal in K and h is the class number of K; then the above relation gives

$$\left\{\frac{\omega}{\mathfrak{p}}\right\} = 1$$

for any $\mathfrak{p}$ in K prime to l. From this it follows† that ω is the lth power of the number in K; whence η is also an lth power. We may then state the following theorem.

If an algebraic field K contains a cyclotomic field $k(\zeta)$, $\zeta = e^{2i\pi/l}$, and η is a primary unit in the former field, then η is the lth power of a unit in K provided the class number of K is prime to l.

We now observe that super-cyclic fields exist in which the class number is prime to l. Such a field is a Kummer field defined by ζ and $(\sigma)^{1/l}$, where σ is a unit in $k(\zeta)$ which is not primary. The class number of such a field is prime to l, provided‡ the class number of $k(\zeta)$ is prime to l.

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* Takagi, Journal of the College of Sciences, Tokyo, vol. 44 (1922), p. 26.

† Hilbert, Werke, vol. 1, p. 276.

‡ Pollaczek, Mathematische Zeitschrift, vol. 21 (1924) p. 6.

3. *The Unit E_n not an l^2th Power.* We now consider the units in $k(\zeta)$ when the class number of this field is not prime to l. In this case the integer c in (2) might be divisible by l; in particular one of the E's may be the lth power of the unit in $k(\zeta)$.

We shall now show that if

$$r^{l-1} \not\equiv 1 \pmod{l^2},$$

then

$$E_n \neq \rho^{l^2},$$

where ρ is in $k(\zeta)$. Assuming an equality of this type, and using the same method by which, in a previous paper by the writer, the relations (3) and (3a) were handled,* we obtain the following identity in e^v:

$$E_n{}^{l-1}(e^v) = (\rho(e^v))^{l^2(l-1)} + X(e^v)(e^{vl} - 1) + lj\frac{e^{vl} - 1}{e^v - 1},$$

where j is a rational integer and $X(e^v)$ is a polynomial in e^v with rational integral coefficients. In this expression, taking logarithms and differentiating $2l$ times, we obtain, using relations (4) and (4a) of the paper last mentioned (p. 620) for $n \neq 1$,

$$\frac{r^{(l-1)(l-n)} - 1}{r^{2l-2n} - 1} \frac{B_l}{2l}(r^{2l} - 1) \equiv 0 \pmod{l^2}.$$

Now, since

$$r^{l-1} \not\equiv 1 \pmod{l^2},$$

then

$$r^{(l-1)(l-n)} - 1$$

is divisible by l but not by l^2, which gives a contradiction since $(r^{2l} - 1)$ and B_l/l are prime to l.

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* Loc. cit., p. 617.