	27	14	1	10	9	23	5	19	18
(2)	13	3	26	8	22	12	21	17	4
	2	25	15	24	11	7	16	6	20
	27	14	1	10	9	23	5	19	18
(3)	11	7	24	6	20	16	25	15	2
	4	21	17	26	13	3	12	8	22
	27	13	2	10	8	24	5	21	16
(4)	11	9	22	6	19	17	25	14	3
	4	20	18	26	15	1	12	7	23

Each of these normalized cubes represents a group of 1296, so that there are altogether 5,184 magic cubes of order 3.

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EXTENSION OF RANGE OF FUNCTIONS*

BY E. J. McSHANE

A well known and important theorem of analysis states that a function $f(x)$ which is continuous on a bounded closed set E can be extended to the entire space, preserving its continuity. Let us consider a metric space S and a function $f(x)$ defined and possessing a property P on a subset E of S. We shall for the sake of brevity say that *$f(x)$ can be extended to S preserving property P*, if there exists a function $\phi(x)$, defined and possessing property P on all of S, which is equal to $f(x)$ for all x on E. Our present object is to establish an easily proved theorem which both includes the classical theorem stated above, and also shows that functions satisfying a Lipschitz or Hölder condition on an arbitrary set E can be extended to S preserving the Lipschitz or Hölder condition. An advantage of the present procedure is that it yields an explicit formula for the extension.†

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† After this paper was submitted for publication, the author found that Hassler Whitney had already indicated a simple proof that a function continuous on a bounded closed set can be extended to be continuous on all space, the method of extension being almost identical with the present one. (H. Whitney, Transactions of this Society, vol. 36 (1934), footnote on p. 63).

In order to exhibit the simplicity of the method we first consider the special case of Lipschitzian functions. In our metric space S we denote the distance of points x_1, x_2 by $\|x_1, x_2\|$.

THEOREM 1. *Let the real-valued function $f(x)$ be defined on a subset E of the metric space S and satisfy the Lipschitz condition*

$$|f(x_1) - f(x_2)| \leqq M\|x_1, x_2\| \tag{1}$$

on E. Then $f(x)$ can be extended to S preserving the same Lipschitz condition.

For every x of S we define $\phi(x)$ to be the least upper bound of $f(\bar{x}) - M\|\bar{x}, x\|$ for all $\bar{x}$ of E: $\phi(x) = \overline{B}(f(\bar{x}) - M\|\bar{x}, x\|)$. If x is in E, we have $\phi(x) = f(x)$; for by (1), $f(\bar{x}) - M\|\bar{x}, x\| \leqq f(x)$, and this upper bound $f(x)$ is attained for $\bar{x} = x$. Also, $\phi(x)$ satisfies the Lipschitz condition. For let x, x' be any two points of S, and suppose to be specific that $\phi(x') \geqq \phi(x)$. Then, unless $\phi(x) = \phi(x') = \infty$,

$$\begin{aligned} 0 \leqq \phi(x') - \phi(x) &= \overline{B}(f(\bar{x}) - M\|\bar{x}, x'\|) - \overline{B}(f(\bar{x}) - M\|\bar{x}, x\|) \\ &\leqq \overline{B}[(f(\bar{x}) - M\|\bar{x}, x'\|) - (f(\bar{x}) - M\|\bar{x}, x\|)] \\ &= \overline{B}M(\|\bar{x}, x\| - \|\bar{x}, x'\|) \leqq M\|x, x'\|. \end{aligned}$$

Therefore, unless $\phi(x) = \phi(x') = \infty$, we have

$$|\phi(x) - \phi(x')| \leqq M\|x, x'\|. \tag{2}$$

In particular, if x is arbitrary and x' is in E, $\phi(x')$ is finite; then (2) holds. Hence, by (2), $\phi(x)$ is always finite and (2) always holds.

Before proceeding to the general case we prove a lemma concerning moduli of continuity. If $f(x)$ be defined on E, we define its *least modulus of continuity* $\omega_0(t)$ by the relation

$$\omega_0(t) = \overline{B}|f(x) - f(\bar{x})| \textit{ for all } x, \bar{x} \textit{ of } E \textit{ such that } \|x, \bar{x}\| \leqq t.$$

We say that $\omega(t)$ *is a modulus of continuity of the function* $f(x)$, if $\omega(t) \geqq \omega_0(t)$; that is, if the inequality $|f(x) - f(\bar{x})| \leqq \omega(t)$ holds for all points x, $\bar{x}$ of E such that $\|x, \bar{x}\| \leqq t$. Clearly $\omega_0(t)$ is non-negative and monotonic increasing; and by definition, $f(x)$ is uniformly continuous on E if $\omega_0(t)$ tends to zero with t.

LEMMA. *If $\omega_0(t)$ is defined and non-negative for $t \geqq 0$, and there exist constants h, k such that $\omega_0(t) \leqq ht + k$, then there exists a func-*

tion $\omega(t) \geqq \omega_0(t)$ *which is continuous and concave* for all* $t>0$. *Moreover, if* ω_0 *tends to* 0 *with* t, *we may require* $\omega(t)$ *to do the same.*

In the (t, u) plane we consider the totality of all lines $u=at+b$ for which the inequality $\omega_0(t) \leqq at+b$ holds for all t. For each such line the region $t \geqq 0$, $u \leqq at+b$ is a convex region; hence the common part Π of all these regions is convex. Its upper boundary is a concave curve $u=\omega(t)$; and since the curve $u=\omega_0(t)$ lies in Π, we have $\omega(t) \geqq \omega_0(t)$. On any interval $0 \leqq t \leqq t_0$, the function $\omega(t)$ is bounded; being concave, it is therefore continuous. Suppose further that $\omega_0(t) \rightarrow 0$; then to every $\epsilon>0$ there corresponds a δ such that $0 \leqq \omega_0(t) \leqq \epsilon$ for $t \leqq \delta$. We can easily find a linear function $u=at+\epsilon$, $a>0$, such that $at+\epsilon>ht+k \geqq \omega_0(t)$ for $t>\delta$. Then $\omega(t)<at+\epsilon$, so that $0 \leqq \lim_{t\rightarrow 0}\omega(t) \leqq \epsilon$. This being true for every $\epsilon>0$, the limit of $\omega(t)$ must be zero.

As examples of functions $f(x)$ for which $\omega_0(t)$ satisfies the hypotheses of the lemma, we have the following.

(a) All bounded uniformly continuous functions. For if $|f| \leqq M$, then $\omega_0(t) \leqq 2M$.

(b) All functions satisfying a Hölder condition

$$|f(x_1)-f(x_2)| \leqq M\|x_1, x_2\|^{\alpha}, \qquad (0<\alpha \leqq 1).$$

For then $\omega_0(t) \leqq Mt^{\alpha} \leqq M(1+t)$.

THEOREM 2. *If* $f(x)$ *is a real function defined on a subset* E *of a metric space* S, *and* $f(x)$ *has a modulus of continuity* $\omega(t)$ *which is concave for* $t \geqq 0$ *and which approaches zero with* t, *then* $f(x)$ *can be extended to* S *preserving the modulus of continuity* $\omega(t)$.

As a first remark, we observe that $\omega(t)$ is monotonic increasing; otherwise it would be negative for sufficiently large t, which is absurd. We now define

$$\phi(x) \equiv \overline{B}(f(\bar{x})-\omega(\|\bar{x}, x\|)),$$

where x ranges over E. For all x of E we have $\phi(x)=f(x)$, because by hypothesis $f(\bar{x})-\omega(\|\bar{x}, x\|) \leqq f(x)$, and this upper bound $f(x)$ is attained for $\bar{x}=x$.

Let now x and x' be any two points of S for which $\phi(x)$ and $\phi(x')$ are not both infinite, and suppose to be specific that

* Concave downwards.

$\phi(x) \geqq \phi(x')$. Then, remembering the monotoneity and concavity of $\omega(t)$, we see that

$$\begin{aligned} 0 \leqq \phi(x) - \phi(x') &= \overline{B}(f(\bar{x}) - \omega(\|\bar{x}, x\|)) - \overline{B}(f(\bar{x}) - \omega(\|\bar{x}, x'\|)) \\ &\leqq \overline{B}[(f(\bar{x}) - \omega(\|\bar{x}, x\|)) - (f(\bar{x}) - \omega(\|\bar{x}, x'\|))] \\ &\leqq \overline{B}[\omega(\|\bar{x}, x\| + \|x, x'\|) - \omega(\|\bar{x}, x\|)] \\ &\leqq \omega(\|x, x'\|). \end{aligned}$$

Hence, unless $\phi(x) = \phi(x') = \infty$, the inequality

$$|\phi(x) - \phi(x')| \leqq \omega(\|x, x'\|) \tag{3}$$

holds. If we choose x' in E, $\phi(x')$ is finite, so that (3) holds for all x. This implies that $\phi(x)$ is always finite, so that (3) holds for all x and x' of S without exception.

COROLLARY 1. *If $f(x)$ satisfies on E a Lipschitz or Hölder condition*

$$|f(x_1) - f(x_2)| \leqq M\|x_1, x_2\|^{\alpha}, \qquad (0 < \alpha \leqq 1),$$

then $f(x)$ can be extended to S preserving the Lipschitz or Hölder condition.

For in Theorem 2 we can take $\omega(t) = Mt^{\alpha}$.

COROLLARY 2. *If $f(x)$ is bounded and uniformly continuous on E, it can be extended to S preserving the uniform continuity and the bounds.*

For by the lemma, $f(x)$ has a modulus of continuity $\omega(t)$ satisfying the hypotheses of Theorem 2. We can therefore find a function $\overline{\phi}(x)$, defined and with modulus of continuity $\omega(t)$ on S and coinciding with $f(x)$ on E. If m, M are, respectively, the lower and upper bounds of $f(x)$ on E, we define $\phi(x) = \overline{\phi}(x)$ where $m \leqq \overline{\phi}(x) \leqq M$, $\phi(x) = M$ where $\overline{\phi}(x) > M$, $\phi(x) = m$ where $\overline{\phi}(x) < m$. Then $\phi(x)$ has bounds m and M, and $\omega(t)$ serves as a modulus of continuity for $\phi(x)$; and on E we have $\phi = \overline{\phi} = f$.

As a special case of Corollary 2, a function $f(x)$ defined and continuous on a bounded closed subset E of n-dimensional euclidean space S_n can be extended to S_n, preserving continuity and the bounds. For $f(x)$ is necessarily bounded and uniformly continuous on E.

COROLLARY 3. *If $f(x)$ is uniformly continuous on E, it can be extended to be continuous on S.*

If x' is a limit point of E, $f(x)$ has a unique limit as x approaches x', because of the uniform continuity of f. Hence we can extend f at once to the closure $\overline{E}$ of the set E, preserving uniform continuity. If we define $p(x)$ to be the distance of x from the set $\overline{E}$, then $p(x)=0$ for x in $\overline{E}$ and $p(x)>0$ for all other x. Let us now define $f_1(x)=(2/\pi)$ arc $\tan f(x)$; then $|f_1|<1$ for all x in $\overline{E}$. Since $|f_1(x)-f_1(x')|<|f(x)-f(x')|$, it is clear that f_1 is uniformly continuous on $\overline{E}$. By Corollary 2 there exists a function $\overline{\phi}(x)$, continuous on S, coinciding with f_1 on $\overline{E}$, and such that $|\overline{\phi}|\leqq 1$. If we define $\phi(x)=\overline{\phi}(x)(1+p(x))^{-1}$, then for all x in $\overline{E}$ we have $|\phi(x)|=|\overline{\phi}(x)|<1$, while for all other x we have $|\phi(x)|<|\overline{\phi}(x)|\leqq 1$. Hence the inequality $|\phi(x)|<1$ holds for all x. The function $\psi(x)=\tan\ (\pi\phi(x)/2)$ is then the desired extension of f. For on $\overline{E}$ we have $\phi(x)=\overline{\phi}(x)=f_1(x)$, so that $\psi(x)=f(x)$; and since ϕ is continuous and less than 1 in absolute value, $\psi(x)$ is continuous for every x.

If we compare Corollary 3 with Theorem 2 (and the lemma), we see that it has weaker hypotheses, since we do not assume that $\omega_0(t)$ is less than a linear function $ht+k$, and it also has a weaker conclusion, since the extension of $f(x)$ is not necessarily uniformly continuous. As a matter of fact, if S is a linear space it can be shown with little difficulty that in order for $f(x)$ to be extensible to a uniformly continuous function S it is necessary that $\omega_0(t)$ be less than some linear function.

Before proceeding to Corollary 4 we introduce some new notation. We denote by $K(x, r)$ the sphere with center x and radius r; that is, the set of all x' such that $\|x', x\|<r$. The surface of the sphere we denote by $K^*(x, r)$; this is the set of all x for which $\|x', x\|=r$. The sum of $K(x, r)$ and $K^*(x, r)$ is $\overline{K}(x, r)$. These are all metric spaces, if not vacuous.

COROLLARY 4. *If $f(x)$ is bounded and uniformly continuous on every sphere $K(x, r)$ of S, it can be extended to be continuous on S, the extension being bounded and uniformly continuous on every sphere $K(x, r)$.*

As in Corollary 3, we may assume that the range of definition E is closed. We choose a point x_0 of S, and denote by K_n the sphere $K(x_0, n)$. For each n we can extend $f(x)$ (considered as a function on $E\cdot K_{n+1}$ alone) to be bounded and uniformly continuous on $E\cdot K_{n+1}+K_n^*$, by Corollary 2. For each n, this extended f is bounded and uniformly continuous on the set $E\cdot(\overline{K}_n-K_{n-1})+K_{n-1}^*+K_n^*$, and hence can be extended to be bounded and uniformly continuous on $\overline{K}_n-\overline{K}_{n-1}$. The extensions of f thus obtained piece together to form a function $\phi(x)$ which is bounded and uniformly continuous on each K_n and continuous on all of S. Every sphere $K(x, r)$ lies in some K_n, hence $\phi(x)$ is bounded and uniformly continuous on $K(x, r)$.

As a consequence of Corollary 4, *every function defined and continuous on a closed set E of euclidean n-space S_n can be extended to be continuous on S_n.*

Corollary 4 does not include Corollary 3. Suppose for example that S is a Hilbert space and $x_1, x_2, \cdots$ is a normed orthogonal set. If we assign $f(x)$ any values on the x_i, it can be extended (by Corollary 3) to be continuous on S, because $\omega_0(t)=0$ for $0\leqq t<2^{1/2}$. But the hypotheses of Corollary 4 are not satisfied unless the values $f(x_i)$ are bounded.

A function $f(x, y)$, defined on a set E in the (x, y) plane and absolutely continuous in the sense of Tonelli, can not always be extended even to be of limited total variation, not even if E be an open set plus its boundary. For let I_n denote the interval $2^{-2n-1}\leqq x\leqq 2^{-2n}$, $(n=0, 1, \cdots)$, and let E be the set (x in $\sum I_n$, $0\leqq y\leqq 1$) plus the intervals $x=0$, $0\leqq y\leqq 1$. If x is in I_n, we define $f(x, y)=(-1)^n(n+1)^{-1}$; for $x=0$, we set $f(x, y)=0$. This function is continuous on E and absolutely continuous in the sense of Tonelli in the interior of E. But any function $\phi(x, y)$ defined on the unit square Q: $0\leqq x\leqq 1$, $0\leqq y\leqq 1$, and coinciding with f on E would fail to have limited total variation, for on any abscissa through Q the function would pass through the values $1, -1/2, +1/3, -1/4, \cdots$. It does not help to require E to be simply connected. For to the set E above let us add the part of Q below the line $y=x$, and in each of the trapezoids thus added let us define $f(x, y)$ to be linear in x and joining continuously to the values already assigned on E. The new set E_1 consists of a simply connected open set plus its boundary; $f(x, y)$ is absolutely continuous, and still any extension to Q must fail to have limited total variation.

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