and curvature types of triply infinite families of curves. The reresults are summarized in the following table; the two types compared are named in the left-hand column; their intersection is identified in the center; and the number, that is, the infinitude, of (projectively different) common families is given at the right.

Dynamical Sectional:	Special central fields or General cones	$\infty^{f(1)}$
Dynamical Curvature:	Any central field	$\infty^{f(2)}$
Sectional Curvature:	General cones and Quadric surfaces	$\infty^{f(1)+2}$

The 2 in the exponent of ∞ refers of course to two arbitrary constants, while (according to a notation which I proposed in this Bulletin in 1912, in a review of Riquier's treatise on partial differential equations) $f(1)$ means an arbitrary function of one independent variable, and $f(2)$ an arbitrary function of two independent variables.

COLUMBIA UNIVERSITY

ON NEVANLINNA'S WEAK SUMMATION METHOD†

BY A. F. MOURSUND

1. *Introduction.* Our principal object in this note is to discuss the function

$$\rho_n(\beta) = \frac{2}{\pi}\int_0^{\pi/2}\left|\int_0^1 \beta(\log C)^\beta(1-t)^{-1}(\log C/(1-t))^{-\beta-1} \times\frac{\sin(2nt+1)s}{\sin s}dt\right|ds, \tag{1}$$

which, for $\beta>0$ and the "dummy" constant $C\geqq e^{\beta+1}$, plays a role in the theory of summation of Fourier series by Nevanlinna's weak method‡ analogous to the role the Lebesgue constants

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‡ F. Nevanlinna, *Über die Summation der Fourier'schen Reihen und Integrale*, Översikt av Finska Vetenskaps-Societetens Förhandlingar, vol. 64 (1921–22), A, No. 3, 14 pp. A. F. Moursund, *On the Nevanlinna and Bosanquet-Linfoot summation methods*, Annals of Mathematics, (2), to appear.

$$(2)\qquad \rho_n = \frac{2}{\pi}\int_0^{\pi/2}\left|\frac{\sin(2n+1)s}{\sin s}\right| ds, \quad (n = 0, 1, \cdots),$$

play in the theory of convergence of such series.† Nevanlinna's weak method is essentially the same as the Bosanquet-Linfoot method of zero order.‡

Our principal results concerning the function $\rho_n(\beta)$ are given by the following theorems.

THEOREM 1. *For each* $n \geqq 0$, *the function* $\rho_n(\beta) \rightarrow \rho_n$ *as* $\beta \rightarrow 0$.

THEOREM 2. *When* $\beta > 1$, *the function* $\rho_n(\beta)$ *is uniformly bounded with respect to n for all* $n \geqq 0$.

THEOREM 3. *For* $0 < \beta < 1$,

$$\rho_n(\beta) = \frac{4}{\pi^2}\frac{(\text{llog } C)^\beta}{1-\beta}(\log n)^{1-\beta} + O(1),$$

and

$$\rho_n(1) = \frac{4}{\pi^2}\log C \log\log n + O(1).$$

2. *Nevanlinna's Weak Summation Method.* Applied to the Fourier series generated by a Lebesgue integrable function $f(x)$, Nevanlinna's weak method consists in forming from the well known expression for the sum of n terms of the series the N_β transform

$$(3)\quad N_\beta S_n(x) = \frac{1}{2\pi}\int_0^1 N_\beta(t)dt\int_{-\pi}^{\pi} f(s)\,\frac{\sin(2nt+1)(x-s)/2}{\sin(x-s)/2}\,ds,$$

where

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† L. Fejér, *Lebesguesche Konstanten und Divergente Fourierreihen*, Journal für Mathematik, vol. 138 (1910), pp. 22–53. Fejér shows in that paper that $\rho_n \sim (4/\pi^2)\log n + O(1)$. T. H. Gronwall, *Über des Lebesgueschen Konstanten bei den Fourierschen Reihen*, Mathematische Annalen, vol. 72 (1912), pp. 244–261. G. Szegö, *Über die Lebesgueschen Konstanten bei den Fourierschen Reihen*, Mathematische Zeitschrift, vol. 9 (1921), pp. 163–166.

‡ L. S. Bosanquet and E. H. Linfoot, *On the zero order summability of Fourier series*, Journal of the London Mathematical Society, vol. 6 (1931), pp. 117–126. L. S. Bosanquet and E. H. Linfoot, *Generalized means and the summability of Fourier series*, Quarterly Journal of Mathematics (Oxford Series), vol. 2 (1931), pp. 207–229. Moursund, loc. cit.

$$N_\beta(t) = \beta(\log C)^\beta(1-t)^{-1}(\log C/(1-t))^{-\beta-1},$$

with $\beta > 0$ and $C \geqq e^{\beta+1}$, and in taking the limit

$$\lim_{n\to\infty} N_\beta S_n(x). \tag{4}$$

Bosanquet and Linfoot have given an example, which will serve equally well for Nevanlinna's weak method, of a continuous function $f(x)$ whose Fourier series diverges at $x=0$ when summed by their zero order method with $0<\beta\leqq 1$.† For $\beta>1$, Nevanlinna's method, and consequently the Bosanquet-Linfoot zero order method, is Lebesgue effective.‡

3. *The Function* $\rho_n(\beta)$. Upon changing the order of integration in (3), we see that, for the values of β for which the function

$$\rho_n(\beta, x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\int_0^1 N_\beta(t)\frac{\sin(2nt+1)(x-s)/2}{\sin(x-s)/2}dt\right| ds \tag{5}$$

becomes infinite with n, functions continuous on $(-\pi, \pi)$ can be constructed for which (i) the limit (4) does not exist at the point x, (ii) the limit (4) exists at x but does not exist uniformly in any neighborhood of that point.§ Setting $x=0$ in (5) we obtain, after slight simplification, the function $\rho_n(\beta)$ defined in the introduction.

4. *Lemmas.* In the statements and proofs of our lemmas and theorems, $\beta>0$, $C\geqq e^{\beta+1}$, and $n\geqq 0$ unless otherwise stated. Proofs which the reader can readily supply are merely indicated or omitted entirely.

LEMMA 1. *For* $2ns\geqq\pi$

$$\left|\int_0^1 N_\beta(t)\left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} 2nst\,dt\right| \leqq \int_{1-\pi/(2ns)}^1 N_\beta(t)dt = (\log C)^\beta(\log 2Cns/\pi)^{-\beta}.\|$$

† Loc. cit., first paper.

‡ A. F. Moursund, *On a method of summation of Fourier series*, Annals of Mathematics, (2), vol. 33 (1932), pp. 773–784.

§ E. W. Hobson, *The Theory of Functions of a Real Variable*, 2d ed., vol. 2 Chapter 7.

‖ See Moursund, second loc. cit., pp. 779–780. Lemma 5.1 holds for $N_\beta(t)$ is non-negative and monotone increasing on (0, 1).

LEMMA 2. $\rho_n(\beta) = \rho_n^*(\beta) + O(1)$, *where*

$$\rho_n^*(\beta) = \frac{2}{\pi}\int_0^{1/2} \frac{1}{s}\left| \int_0^1 N_\beta(t) \sin 2nst\, dt \right| ds, \tag{6}$$

and the $O(1)$ *terms are uniformly bounded with respect to* β, C, *and* n.†

LEMMA 3. *For* $v > 0$,

$$\int_0^\infty \left(\frac{\log C}{v} + 1 + t\right)^{-\beta-1} \cos e^{-tv}\, dt = \frac{1}{\beta} + O\left(\frac{1}{v}\right).$$

PROOF. We write

$$\begin{aligned} \frac{1}{\beta} - \int_0^\infty &= \int_0^\infty \left[(1+t)^{-\beta-1} - \left(\frac{\log C}{v} + 1 + t\right)^{-\beta-1} \cos e^{-tv}\, dt \right] \\ &\leqq \int_0^\infty \left\{ \left[(1+t)^{-\beta-1} - \left(\frac{\log C}{v} + 1 + t\right)^{-\beta-1} \right] \right. \\ &\qquad \left. + \left(\frac{\log C}{v} + 1 + t\right)^{-\beta-1} e^{-2tv}/2 \right\} dt \\ &\leqq (\beta+1) \int_0^\infty dt \int_0^{\log C/v} (1+t+u)^{-\beta-2}\, du + \int_0^\infty e^{-2tv}/2\, dt \\ &\leqq (\beta+1) \frac{\log C}{v} \int_0^\infty (1+t)^{-\beta-2}\, dt + \frac{1}{4v} = \frac{\log C}{v} + \frac{1}{4v}. \end{aligned}$$

LEMMA 4. *For* r *sufficiently large*,

$$\int_0^1 N_\beta(1-t) \cos rt\, dt = \beta(\log C)^\beta \{ (\log r)^{-\beta}/\beta + O(\log r)^{-\beta-1} \} > 0,$$

$$\int_0^1 N_\beta(1-t) \sin rt\, dt = O[(\log r)^{-\beta-1}].$$

PROOF. The lemma, except for the term $O[(\log r)^{-\beta-1}]$ in the first part, follows immediately from a theorem concerning Fourier coefficients due to U. S. Haslam-Jones.‡ Upon inspection of

† It can be shown by using Lemma 1 that the $O(1)$ terms are $o(1)$ as $n \to \infty$.

‡ U. S. Haslam-Jones, *A note on the Fourier coefficients of unbounded functions*, Journal of the London Mathematical Society, vol. 2 (1927), pp. 151–154 (Theorem 2).

the proof given by Haslam-Jones, the reader will see how, with the aid of Lemma 3, our result can be obtained.

LEMMA 5.

$$| \sin r | = \frac{2}{\pi} - \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\cos 2mr}{4m^2 - 1} \cdot †$$

LEMMA 6. *For* $M \geqq e$ *and* $n \geqq M$,

$$\sum_{m=1}^{\infty} \frac{1}{4m^2 - 1} \int_M^n \frac{1}{r} (\log r)^{-\beta} \cos 2mr \, dr \leqq \frac{1}{2M}.$$

PROOF. It can be shown by means of the second mean value theorem that each of the integrals $\leqq 1/M$. The lemma follows, because

$$\sum \frac{1}{4m^2 - 1} = \frac{1}{2}.$$

5. *Proof of Theorems.* In this paragraph we prove our theorems.

PROOF OF THEOREM 1. Integrating by parts, we have

$$\rho_n(\beta) = \frac{2}{\pi} \int_0^{\pi/2} \left| 1 + (\log C)^\beta \frac{2ns}{\sin s} \int_0^1 (\log C/(1 - t))^{-\beta} \times \cos (2nt + 1)s \, dt \right| ds,$$

and the theorem follows by elementary theorems concerning limits.

PROOF OF THEOREM 2. When $n \leqq \pi$, we have at once $\rho_n^*(\beta) \leqq 2$; and when $n > \pi$, we have for $\beta > 1$, using Lemma 1,

$$\rho_n^*(\beta) \leqq O(1) + \frac{2}{\pi} (\log C)^\beta \int_{\pi/(2n)}^{1/2} \frac{1}{s} (\log 2Cns/\pi)^{-\beta} \, ds = O(1).$$

The theorem follows by Lemma 2.

PROOF OF THEOREM 3. By Lemma 2,

$$\rho_n(\beta) = \rho_n^*(\beta) + O(1).$$

† Szegö, loc. cit., uses this Fourier series expansion in obtaining his formula for ρ_n.

Using Lemmas 4, 5, and 6, we have, for a fixed sufficiently large M and $n>M$,

$$
\begin{aligned}
\rho_n^*(\beta) &= \frac{2}{\pi}\left[\int_0^{M/(2n)} + \int_{M/(2n)}^{1/2}\right] \frac{1}{s}\left|\int_0^1 N_\beta(t)\sin 2nst\,dt\right| ds \\
&= O(1) + \frac{2}{\pi}\int_M^n \frac{1}{r}\left|\int_0^1 N_\beta(t)\sin rt\,dt\right| dr \\
&= \frac{2}{\pi}\int_M^n \frac{1}{r}\left|\sin r\int_0^1 N_\beta(1-t)\cos rt\,dt\right. \\
&\qquad \left. -\cos r\int_0^1 N_\beta(1-t)\sin rt\,dt\right| dr + O(1) \\
&= \frac{2}{\pi}\int_M^n \frac{|\sin r|}{r}\,dr \int_0^1 N_\beta(1-t)\cos rt\,dt + O(1) \\
&= \frac{2\beta(\log C)^\beta}{\pi}\left\{\frac{2}{\pi}\int_M^n \frac{1}{r}\cdot\frac{(\log r)^{-\beta}}{\beta}\,dr\right. \\
&\qquad - \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{1}{4m^2-1}\int_M^n \frac{(\log r)^{-\beta}}{\beta}\cos 2mr\,dr \\
&\qquad \left. + \int_M^n \frac{|\sin r|}{r} O\left[(\log r)^{-\beta-1}\right]dr\right\} + O(1) \\
&= \frac{4}{\pi^2}(\log C)^\beta \int_M^n \frac{1}{r}\cdot(\log r)^{-\beta}\,dr + O(1).
\end{aligned}
$$

The lemma follows when we carry out the integration.

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