[(1), (2), 11.6] $\sim\sim\sim\sim p < p$ (3);

[(3), 11.02] $\sim\Diamond(\sim\sim\sim\sim p \sim p)$ (4);

$\left[\text{Th. } 2, \dfrac{\sim\sim\sim\sim p}{p}, \dfrac{\sim p}{q}\right]$

$\sim\sim\sim\sim p \sim p = \sim p \sim\sim\sim\sim p$ (5);

[(4), (5)] $\sim\Diamond(\sim p \sim\sim\sim\sim p)$ (6);

[(6), 11.02] $\sim p < \sim\sim\sim p$ (7);

$\left[\text{Th. } 3, \dfrac{\sim p}{p}\right]$ $\sim\sim\sim p < \sim p$ (8);

[(7), (8), 11.03] $\sim p = \sim\sim\sim p$ (9);

[Th. 1, 11.02] $\sim\Diamond(p \sim p)$ (10);

[(10), (9)] $\sim\Diamond(p \sim\sim\sim p)$ (11);

[(11), 11.02] $p < \sim\sim p$,

which was to be proved.

Theorem 4 is Lewis' postulate 11.5.

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NOTES ON THE AIRY STRESS FUNCTION

BY J. H. A. BRAHTZ

1. *The Expressions for the Components of Stress.* Problems in plane stress and plane strain are usually best treated by use of the Airy stress function even when body forces exist. It has been customary to express the stresses as follows:

$$(1)\quad \sigma_x = \frac{\partial^2 A}{\partial y^2} - C_1 x, \qquad \sigma_y = \frac{\partial^2 A}{\partial x^2} - C_2 y, \qquad \tau_{xy} = -\frac{\partial^2 A}{\partial x \partial y},$$

where C_1 and C_2 are components of the uniform body force C per unit volume acting at an angle β with the x-axis. Hence

$$C_1 = C\cos\beta, \qquad C_2 = C\sin\beta.$$

It will be found that an arbitrary function $A(x, y)$ will satisfy the equations of equilibrium for a rectangular element $dx\,dy$:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + C_1 = 0, \qquad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + C_2 = 0. \tag{2}$$

In order to be compatible with the generalized Hooke's law, A must be a solution of the equation

$$\nabla^4 A \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 A = 0. \tag{3}$$

In polar coordinates the stresses are defined by the equations

$$\begin{aligned} \sigma_r &= \frac{\partial^2 F}{r^2 \partial \theta^2} + \frac{1}{r}\frac{\partial F}{\partial r} - Cr\cos(\theta - \beta), \\ \sigma_\theta &= \frac{\partial^2 F}{\partial r^2} - Cr\cos(\theta - \beta), \\ \tau_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{\partial F}{r\partial \theta}\right). \end{aligned} \tag{4}$$

These satisfy identically the equations of equilibrium for an element $r\,dr\,d\theta$:

$$\begin{aligned} &\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_r) + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{\sigma_\theta}{r} + C\cos(\theta - \beta) = 0, \\ &\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} - C\sin(\theta - \beta) = 0. \end{aligned} \tag{5}$$

Again F must be a solution of the equation

$$\nabla^4 F \equiv \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{r^2\partial \theta^2}\right)^2 F = 0. \tag{6}$$

If A and F have been completely determined for the same set of boundary conditions, with the same origin and x-axis, it can be shown that

$$F = A + \frac{1}{6}C_1 x^3 + \frac{1}{6}C_2 y^3. \tag{7}$$

It is convenient to have the same Airy function in both sets of coordinates. This is possible only if the expressions for the stresses in rectangular coordinates are changed to the following form:

$$(8)\quad \sigma_x = \frac{\partial^2 F}{\partial y^2} - C_1 x - C_2 y, \quad \sigma_y = \frac{\partial^2 F}{\partial x^2} - C_1 x - C_2 y, \quad \tau = -\frac{\partial^2 F}{\partial x \partial y}.$$

In this manner the stresses may be obtained in cartesian as well as polar coordinates by the same function F, and the danger of making serious errors when body forces exist is eliminated.

2. *Convenient Forms for Boundary Conditions.* CASE 1. *No body forces exist, that is,* $C = C_1 = C_2 = 0$. We have

$$(9)\quad d\frac{\partial F}{\partial x} = \frac{\partial^2 F}{\partial x^2} dx + \frac{\partial^2 F}{\partial x \partial y} dy, \qquad d\frac{\partial F}{\partial y} = \frac{\partial^2 F}{\partial x \partial y} dx + \frac{\partial^2 F}{\partial y^2} dy.$$

Integrating along the boundary from A to B in the direction (XY), and substituting the definition (8), we obtain

$$\left[\frac{\partial F}{\partial x}\right]_A^B = \int_A^B \sigma_y dx - \int_A^B \tau\, dy, \quad \left[\frac{\partial F}{\partial y}\right]_A^B = \int_A^B \sigma_x dy - \int_A^B \tau\, dx.$$

But, on the boundary, $\sigma_y dx - \tau_x dy = -Q ds$, $\sigma_x dy - \tau dx = +P ds$, where P and Q are the components of the boundary forces per unit of contour. If the origin is placed at A, it is always possible* to make

$$(10)\qquad (F)_A = \left(\frac{\partial F}{\partial x}\right)_A = \left(\frac{\partial F}{\partial y}\right)_A = 0.$$

Then

$$(11)\quad \frac{\partial F}{\partial x} = -\int_A^B Q ds = -Y, \qquad \frac{\partial F}{\partial y} = +\int_A^B P ds = +X.$$

Again we have

$$(12)\qquad dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy;$$

and, integrating along the boundary from A to B, and making use of (10) and (11), we find

$$(13)\qquad F = M.$$

The quantities X, Y, and M are, respectively, the projection on

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* This question is treated in Riemann-Weber, *Differentialgleichungen der Physik*, vol. 2, but stipulation (10) is not made clear.

the X-axis and Y-axis, and the moment about B, of all boundary forces between A and B. Equation (11) may be replaced by the equation

$$(14)\qquad \frac{\partial F}{\partial n} = \frac{\partial F}{\partial x}\cos\alpha + \frac{\partial F}{\partial y}\sin\alpha = X\sin\alpha - Y\cos\alpha,$$

where α is the angle between the x-axis and the outward normal to the boundary. If point B is reached by going along the boundary in the negative direction, the sign of the quantities must be reversed.

CASE 2. *Only body forces exist*, that is, $X = Y = M = 0$. Again substituting definition (8) with $C \neq 0$ into (9), and making use of (10), we find

$$(15)\qquad \begin{aligned} \frac{\partial F}{\partial x} &= \int_A^B (C_1x + C_2y)dx \equiv R, \\ \frac{\partial F}{\partial y} &= \int_A^B (C_1x + C_2y)dy \equiv S, \end{aligned}$$

or

$$\frac{\partial F}{\partial n} = R\cos\alpha + S\sin\alpha.$$

Likewise, by (12),

$$(16)\quad F = C_1\int_A^B \int xdydy + C_2\int_A^B \int ydxdx + \frac{C_1x^3}{6} + \frac{C_2y^3}{6}.$$

Problems in plane stress or plane strain are thus reduced to the determination of a function F which is a solution of $\nabla^4F = 0$ and satisfies the boundary conditions given by (11) to (16).

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