ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

191. Professor J. L. Walsh: *On the convergence and overconvergence of sequences of polynomials of best approximation.*

In the study of the convergence of sequences of polynomials of best approximation in the sense of least pth powers ($p>0$) to a function analytic on a given closed limited point set as measured by various integrals with non-negative norm functions, characteristic results on overconvergence (i.e., proof of greatest geometric degree of convergence) can be established if the norm function and some negative power of it are integrable. If approximation is measured by a line integral over some rectifiable Jordan curve or finite set of rectifiable Jordan curves or arcs C, it is sufficient if the norm function is integrable, and if on each closed subset of C containing no point of a given reducible set of C some negative power is integrable. (Received March 17, 1934.)

192. Professor J. L. Walsh: *On interpolation by polynomials.*

The following theorem is an extension of results due to Kalmár and Fekete. *Let C be a closed limited point set of the z-plane whose complement K is connected and possesses a Green's function with pole at infinity. Let $w=\Phi(z)$, $\Phi'(\infty)=1$, map K conformally (not necessarily uniformly) onto $|w|>r$. Let the points $\beta_1^{(n)}, \beta_2^{(n)}, \cdots, \beta^{(n)}{}_{n+1}$ $(n=0, 1, 2, \cdots)$ have no limit point exterior to C. If $f(z)$ is analytic on C, and if $p_n(z)$ is the polynomial of degree n which interpolates to $f(z)$ in the points $\beta_k^{(n)}$, then equivalent necessary and sufficient conditions that $p_n(z)$ converge to $f(z)$ uniformly on C for every such $f(z)$ are* $\lim_{n\to\infty}\left|(z-\beta_1^{(n)})\cdot(z-\beta_2^{(n)})\cdots(z-\beta^{(n)}{}_{n+1})^{1/(n+1)}\right| = |\Phi(z)|$ *in K,* $\lim_{n\to\infty}\left[\max\left|(z-\beta_1^{(n)})(z-\beta_2^{(n)})\cdots(z-\beta_{n+1}^{(n)})\right|,\ z \text{ on } C\right]^{1/(n+1)}=r$. *If these conditions are satisfied, the polynomial $p_n(z)$ converges to $f(z)$ on C with the greatest geometric degree of convergence, as does the polynomial $P_n(z)$ of degree n found by interpolation to $f(z)$ in the first $n+1$ of the points $\beta_1^{(0)}, \beta_1^{(1)}, \beta_2^{(1)}, \beta_1^{(2)}, \beta_2^{(2)}, \beta_3^{(2)}, \beta_1^{(3)}, \cdots$. The polynomial $P_n(z)$ is the sum of the first $n+1$ terms of a series of interpolation.* (Received March 17, 1934.)

193. Mr. F. B. Jones: *Concerning locally peripherally separable spaces.*

A space is locally peripherally separable provided that, if P is a point of a region R, there exists in R a domain D containing P such that the boundary of D is separable. The principal theorem established in this paper is as follows: If a connected, locally connected metric space S is locally peripherally separable, then S is completely separable. It follows in particular that a space satisfying Axioms 0, 1, 3, and 5_1 of R. L. Moore's *Foundations of Point Set Theory* is completely separable if metric. (Received March 30, 1934.)

194. Dr. R. C. Hildner: *An inverse problem associated with the Hamilton-Carathéodory function.*

C. Carathéodory (Acta Mathematica, vol. 47) has shown that to the Hamilton function H of a problem of Lagrange there corresponds the entire class of *equivalent* problems of Lagrange. In the present paper an inverse problem connected with what may be called the Hamilton-Carathéodory function $\mathfrak{H}(x, y_1, \cdots, y_n, u_1, \cdots, u_{n+m})$, $\mathfrak{H}(x, y_1, \cdots, y_n, u_1, \cdots, u_n, 0, \cdots, 0) = H(x, y_1, \cdots, y_n, u_1, \cdots, u_n)$, is formulated and solved. Omitting reference to obvious conditions phrased in terms of class properties, it is found that necessary and sufficient conditions for an assigned function $\mathfrak{H}$ to be the Hamilton-Carathéodory function of a problem of Lagrange are that the determinants $\delta = |\partial^2\mathfrak{H}/\partial u_r \partial u_c|$ $(r, c = 1, 2, \cdots, n+m)$ be different from zero and that (possibly after suitably renaming the u_i) every nth order determinant which can be formed from the elements of the matrix of the first n columns of δ and which contains elements from at most $(m-1)$ of the last m rows of δ vanishes identically. The problem of Lagrange associated with an admissible function $\mathfrak{H}$ is exhibited and is found to be *unique*, a result obtained in the case $m=0$ by L. La Paz (this Bulletin, vol. 35, p. 452). (Received March 19, 1934.)

195. Professor H. W. March: *The deflection of a rectangular strip of plywood under a load concentrated along a line through the center.*

The differential equation for the deflection of a sheet or plate of plywood of given structure under a given load is set up. This equation is solved for rectangular strips of plywood, simply supported at the ends, free along the two remaining sides, and loaded uniformly along a line passing through the center of the strip and perpendicular to the free edges. This corresponds to the actual situation in tests to determine the apparent Young's modulus in bending. To determine the circumstances under which the assumption that the strip may be treated as a thin plate is valid and the correction that is to be made when it is not, the strip is also treated as a double cantilever in a state of plane strain. This leads to an expression for the correction as a function of the span-depth ratio. This correction is calculated in terms of the elastic constants of the individual plies for plywood of three, five, seven, and nine plies, the plies being of equal thickness and symmetrically placed with respect to the middle plane.

A strip of isotropic material is also treated as a plate and in a state of plane strain. (Received March 28, 1934.)

196. Professor V. G. Grove: *On a certain correspondence between surfaces in hyperspace.*

Associated with each point on a surface in hyperspace is a certain space, called the two-osculating space of the surface at the point, defined as the ambient space of the osculating planes of all of the curves on the surface through the point. This paper studies the point-to-point correspondence between two surfaces such that the two-osculating spaces of the surfaces at corresponding points coincide. It is found that both surfaces are considerably restricted. In fact in certain cases the lines joining corresponding points form a congruence in the usual sense. In the cases in which the two-osculating spaces of the surfaces are of four dimensions the surfaces must sustain either a conjugate net one of whose component families of curves is a family of plane curves, or a one-parameter family of straight lines. (Received March 17, 1934.)

197. Professor Harry Levy: *Non-Riemannian manifolds in Riemannian space.*

The author investigates the geometry of n congruences of curves in Riemannian space and finds that he is led to a linear connection which is not Riemannian. Parallelism is independent of the path when the angle, at a point, between any two curves of different congruences is the same for all points, and then two directions are parallel if they make equal angles with corresponding curves of the congruences. Further properties of parallelism are obtained, curvature of a curve is defined, and several properties of curvature are given. The geometry here developed is related to conformal geometry by showing that parallelism with respect to a congruence is invariant under conformal transformations of one Riemannian space into another. Conversely, if between two Riemannian spaces there exists a correspondence which preserves parallelism with respect to every ennuple, the correspondence is conformal. (Received March 12, 1934.)

198. Professor Tibor Radó: *On convex functions.*

The functions $f(x)$ considered in this paper are supposed to be continuous and positive in a given open interval $x_1<x<x_2$. If α, β are real numbers, we put $I(f,x,h,\alpha)=\left[\int_{-h}^{h} f(x+\xi)^{\alpha}d\xi\right]^{1/\alpha}/(2h), A(f,x,h,\beta)=\left[(f(x+h)^{\beta}+f(x-h)^{\beta})/2\right]^{1/\beta}$. For $\alpha=0, \beta=0$, I and A are defined by their well-known limits for $\alpha\to 0, \beta\to 0$. The following definitions are used. C denotes the class of all those functions $f(x)$ which are continuous, positive, and convex in $x_1<x<x_2$. $C(\alpha, \beta)$ denotes the class of all those functions $f(x)$ which are continuous and positive in $x_1<x<x_2$, and satisfy an inequality $I(f,x,h,\alpha)\leqq A(f,x,h,\beta)$ for all values of x and h such that $x_1<x-h<x+h<x_2$. $S(\alpha, \beta)$ denotes the set of all couples (α, β) for which $C(\alpha, \beta)>C$, and $S^*(\alpha,\beta)$ denotes the set of all those couples (α, β) for which $C(\alpha,\beta)<C$. These sets $S(\alpha, \beta)$, $S^*(\alpha, \beta)$, and the analogous sets corresponding to concave functions are explicitly determined, and the re-

sults are applied to the theory of convex functions. (Received March 26 1934.)

199. Mr. Solomon Kullback: *The distribution laws of the difference and quotient of variables independently distributed in Pearson type* III *laws.*

Let $u=x-y$ and $v=\log x-\log y$, where the distribution laws of x and y are respectively given by $x^{p-1}e^{-x}/\lfloor\overline{p}$ and $y^{q-1}e^{-y}/\lfloor\overline{q}$. Then by an application of the theory of characteristic functions we are enabled to derive the distribution laws of u and v. The distribution of $w=x/y$ may be obtained from that of v by the transformation $w=e^v$. The result for u in the special case $p=q$ has been obtained otherwise by Pearson, Stouffer, and David in Biometrika (vol. 24, p. 293 ff.). The result for z where $e^v=(n_1/n_2)\cdot e^{2z}$ has been obtained otherwise by R. A. Fisher in Proceedings of the International Mathematical Congress, Toronto (vol. 2 (1924), p. 805 ff.). Although the results are not entirely new they are presented as examples of the general solution of the distribution problem of statistics by means of characteristic functions. (Received March 17, 1934.)

200. Mr. Marshall Hall: *Arithmetic properties of a partition function.*

The sequence $H^*(n)$ which represents the number of partitions of n different things, is the subject of this paper. The sequence is studied by means of its generating function e^{e^x-1} as the coefficient of x^n in e^{e^x-1} is $H^*(n)/n!$. It is proved that $H^*(n+p)\equiv H^*(n+1)+H^*(n)(\text{mod } p)$ where p is any prime. This includes the previously known theorem $H^*(p)\equiv 2(\text{mod } p)$ as a special case. From this congruence follows the periodicity of the residues of $H^*(n)$ modulo p. We have $H^*(n+h)\equiv H^*(n)$ (mod p) where h is some divisor of p^p-1. A slight extension of the method proves the periodicity of the residues of $H^*(n)$ for any modulus. (Received March 19, 1934.)

201. Mr. Solomon Kullback: *A note on the distribution of a certain partial belonging coefficient.*

In a paper in Metron (vol. 7 (1928), No. 3, pp. 3–46) Romanovsky studied the distributions of various coefficients of racial likeness. In particular he considered the first partial belonging coefficient $\alpha_1{}^1$ of Pearson. Romanovsky was unable to find the distribution of $\alpha_1{}^1$ in "manageable form," but was able to determine the moments of the distribution and its limit as the number of observations increased indefinitely. By means of the theory of characteristic functions we are able to find the distribution of $\alpha_1{}^1$ as a closed formula and show that the moments of this distribution and its limiting form agree with those found by Romanovsky in a different manner. A simple transformation will change the distribution to Student's distribution. (Received March 26, 1934.)

202. Dr. Jacob Levitski: *On automorphisms of certain rings.*

In this paper a set M of automorphisms of a ring R in which the "double-chain-condition" on right ideals holds is considered, and it is proved that the quotient-ring K/N, where K is the set of all the elements of R which remain invariant under each automorphism belonging to M, and N is the radical of K, is semi simple. If M is a finite group and R is semi simple, then also K is semi simple. As an application of the results obtained here, a new proof is derived for the theorem which states the semi simplicity of a group-algebra over a field whose characteristic does not divide the order of the group. (Received April 4, 1934.)

203. Professor J. L. Walsh: *On equipotential curves.*

Let R be an infinite region whose boundary B is finite, and let $G(x, y)$ be Green's function (supposed to exist) for R with pole at infinity. Then the normal at an arbitrary point (x_0, y_0) of R to the locus $G(x, y)=G(x_0, y_0)$ when extended from (x_0, y_0) in the sense of decreasing $G(x, y)$ must intersect the smallest convex region containing B. If R is simply connected, this normal must intersect B. This result has obvious extensions to the case that R is not infinite, where the pole of $G(x, y)$ is an arbitrary point of R. (Received April 25, 1934.)

204. Sister Mary Cleophas Garvin, S.N.D.: *On the convergence of a generalized series and the relation of its coefficients to those of the corresponding power series.*

The series considered in this paper, $\sum_{n=1}^{\infty} a_n z^{\lambda n}/(1-z^{\mu n})$, includes those of Lambert, Weierstrass, and Hansen. The regions of convergence for integral values of λ and μ are determined and uniform convergence is established. Under certain conditions placed upon a_n the function represented by the series has the unit circle as a natural boundary. In that part of its region of convergence which lies within the unit circle every series of the given type can be expanded as a power series whose coefficients A_n are the sum of those a_k's whose subscripts satisfy the congruence $n/k \equiv \lambda \pmod{\mu}$. By means of an inversion function the coefficients a_n are likewise obtained in terms of A_k's, so that a power series whose constant term is zero can be expressed as a series of the given type provided μ is a multiple of λ. (Received April 30, 1934.)

205. Dr. A. F. Moursund: *On Nevanlinna's weak summation method.*

In the theory of summation of Fourier series by Nevanlinna's weak summation method, which is essentially the same as the Bosanquet-Linfoot zero order method (for details see the author's paper, *On the Nevanlinna and Bosanquet-Linfoot summation methods*, Annals of Mathematics, vol. 35, April, 1934), the function $\rho_n(\beta)=2/\pi\int_0^{\pi/2}\left|\int_0^1 \beta\ (\log C)^{\beta}\ (1-t)^{-1}\ (\log C/(1-t))^{-\beta-1} \sin(2nt+1)s \cdot (\sin s)^{-1}\, dt\right| ds$, with $\beta>0$ and the "dummy" constant $C \geqq e^{\beta+1}$, plays a role analogous to the role the well known Lebesgue constants ρ_n play in the theory

of convergence of such series. It is shown in this paper that (i) for each $n \geqq 0$, $\rho_n(\beta) \to \rho_n$ as $\beta \to 0$, (ii) for $0 < \beta < 1$, $\rho_n(\beta) = 4/\pi^2 \ (\log C)^\beta/(1-\beta) + O(1)$, (iii) $\rho_n(1) = 4/\pi^2 \log C \cdot \log \log n + O(1)$, and (iv) for $\beta > 1$, $\rho_n(\beta)$ is uniformly bounded with respect to n for $n \geqq 0$. (Received April 30, 1934.)

206. Dr. Ruth G. Mason: *Further properties of ternary continued fractions.*

Formulas are developed connecting the convergents of a periodic ternary continued fraction with those of the fractions of the set obtained by a cyclic permutation of the partial quotient pairs. The product of all the σ_1's of such a set is shown to be ρ, and formulas are developed connecting the σ's of different fractions in the set. Rules are established for the direct formation of any convergent and for the coefficients M and N of the characteristic equation. The periodic fraction $(p_1, q_1; p_2, q_2; \cdots ; p_n, q_n;)$ has a characteristic equation reciprocal to that of $(-q_n, -p_n; \cdots ; -q_2, -p_2; -q_1, -p_1;)$. If a fraction has the property that it and its inverse belong to the same field independently of the order of the partial quotient pairs, then Lehmer's sufficient linear condition on the p's and q's is shown by a new method to be necessary and sufficient. (Received May 4, 1934.)

207. Dr. A. L. Foster: *On the representation of abstract Boolean algebras.*

The question is considered: Can any abstract Boolean algebra be realized as an algebra of classes? No general answer is given, but it is shown that in a large class of these algebras (see abstract No. 208) such a realization is always possible. (Received May 4, 1934.)

208. Dr. A. L. Foster: *Certain classifications of Boolean algebras.*

The author presents examples of Boolean algebras which show that, unlike the case of finite Boolean algebras, the unrestricted postulates for Boolean algebra are not categorical. The basis is laid for certain classifications of Boolean algebras. Apart from their role as a classification guide, the examples are of interest as they include (1) denumerably-infinite algebras, and (2) *modular* Boolean algebras. (Received May 4, 1934.)

209. Dr. A. L. Foster: *Boolean algebra and the abstract theory of algebraic ideals.*

A general theory of *modular* Boolean algebras is formulated, a theory representing a natural parallel to a good portion of the abstract theory of ideals. (Received May 4, 1934.)

210. Professor H. V. Craig: *On a generalized tangent vector.*

The purpose of this paper is to derive from $F(x, x', \cdots , x^m)$ a vector having the cardinal properties of the Finsler covariant tangent vector. (Received May 7, 1934.)

211. Dr. C. H. Dix: *Mechanical invariants of the sweeping-out process.*

We prove the theorem: If a general bounded distribution of positive mass in a closed connected region R is swept-out on a surface S entirely enclosing R in its interior, then the center of gravity and the principal axes are invariants for the sweeping-out transformation. The theorem is still true for more general sets S. (Received May 7, 1934.)

212. Dr. E. J. McShane: *Extension of range of functions.*

Let $f(x)$ be a real function defined as a subset E of a metric space S (in which distance is denoted by $\|\cdot, \cdot\|$). If $f(x)$ has a property P on E, we wish to extend the range of definition of $f(x)$ to the entire space S in such a way that the extended function also has property P on all of S. It is shown that this is possible for a property P which has as particular cases: (1) $f(x)$ satisfies a Lipschitz or Hölder condition on E. (2) $f(x_1)-f(x_2)\leqq\omega(\|x_1, x_2\|)$, where ω is a continuous function which is concave downwards. As corollaries we have the well known theorems that if S be euclidean n-space, a function $f(x)$ continuous on a bounded closed set E can be extended to be uniformly continuous and bounded on S, and if $f(x)$ be continuous on any closed set E it can be extended to be continuous on S. (Received May 7, 1934.)

213. Mr. Marshall Hall: *The exact modular periods of linear recurrences.*

Engstrom, Carmichael, and others have obtained general periods (mod p) for finite linear recurrences. This paper undertakes to answer two questions, hitherto untreated: (1) Are the general periods so found the best possible? and (2) What is the effect on periodicity (mod p) produced by choosing special sets of initial values? These problems are treated by using the linear operator $E(Eu_n=u_{n+1})$, and the theory of double moduli. Both questions are answered: (1) The general periods found are the best possible, as there exists a sequence which has the maximum period predicted; and (2) A special set of initial values affects the periodicity for only a finite number of primes, namely those dividing a certain determinant. (Received May 8, 1934.)

214. Mr. Garrett Birkhoff: *On the automorphisms of Abelian groups.*

A principal series Π is set up in the group A of the automorphisms of an arbitrary Abelian group G. The factor-groups between successive terms of Π are then exhibited collectively as simple functions of the invariants of G. The least normal subgroups of A, together with a few related results, are also given. (Received May 9, 1934.)

215. Mr. Garrett Birkhoff: *On the automorphisms of groups of prime-power order.*

The "commutator-power" subgroup-structure of an arbitrary group P of prime-power order is shown to have an exact parallel in the group of the automorphisms of P. (Received May 9, 1934.)

216. Professor J. H. Roberts: *Sets which are potentially regular relative to a collection of subsets.*

In the present paper the following theorem is proved: Suppose the separable metric space M is potentially regular relative to a collection Z of self-compact subsets of M. Then there exists a biunivalued and continuous transformation T of M into a separable metric space M^* such that (1) if the point P of M is of potential order n relative to Z then for each $e>0$ there is in M^* a domain U containing $T(p)$ and of diameter $<e$, such that the boundary of U is a subset of the sum of the images of n elements of Z, and (2) if every element of Z is the sum of a finite number of connected sets then the property of a compact subset of M to separate two points of M is invariant under T. We have as corollaries a theorem by G. T. Whyburn and one by the author. The notion, *potentially regular relative to Z*, is due to Whyburn. (Received May 9, 1934.)

217. Dr. R. H. Cameron (National Research Fellow): *Linear differential equations with almost periodic coefficients.*

This paper gives various sets of necessary and sufficient conditions that a system (which may or may not be homogeneous) of linear differential equations with almost periodic coefficients should have all of its solutions almost periodic. It also gives sufficient conditions that a given particular solution should be almost periodic; and it can be shown by an example that these conditions are sufficiently weak so that they do not imply that all the solutions be almost periodic. (Received May 12, 1934.)

218. Mr. Garrett Birkhoff: *Transfinite extensions of the Jordan-Hölder theorem.*

The theorem of Jordan-Hölder can be extended to chief and characteristic series which are well-ordered in the direction of increasing subgroups. But there exists an enumerable Abelian group having chief series well-ordered in the direction of decreasing subgroups which do not satisfy this theorem. (Received May 7, 1934.)

ERRATUM

Volume 40, page 234, abstract No. 190 (by Dr. Gordon Pall): the formula in line 10, which was printed as $o_i \equiv o_{i+1} \equiv 0 \pmod{16}$, should read $o_i \equiv o_{i+1} \equiv 0 \pmod{4}$.