him for linear difference equations. The papers by Poincaré in these fields went a long way toward providing a general theory of such linear equations. However, it is only very recently that a general theory of such equations has been developed.*

The last paper of the volume under review contains the second part of the unsuccessful Prize Essay referred to above, and was only published posthumously in the Acta Mathematica in 1923. Here we find a preliminary treatment of the problem as to when an ordinary linear differential equation of the second order with polynomial coefficients and only regular singular points is such that if we write $y_1(x)/y_2(x)=z$ when y_1 and y_2 are two linearly independent solutions, then x is a meromorphic function of z. This problem had been proposed and incompletely solved by Fuchs in 1880, and Poincaré's essay is mainly a critique of Fuchs's work. In this article is to be found the genesis of Poincaré's work in the theory of automorphic functions, which is contained in volume II of his Collected Works.

The reader of volume I will also be very grateful for the careful and highly competent revision which Professor Drach has provided.

G. D. Birkhoff

EISENHART ON CONTINUOUS GROUPS

Continuous Groups of Transformations. By L. P. Eisenhart. Princeton, Princeton University Press, and London, Oxford University Press, 1933. 10+299 pp.

This work is an up-to-date textbook on Lie's theory of continuous groups and on the recent developments of this theory. It is, in contrast to some introductions to the subject, not a book written for a beginner who knows only calculus. It will, however, be enjoyed by every student who knows some existence theorems on differential equations and is familiar with the elements of the tensor symbolism. He will not be bored through a considerable part of the book by the almost traditional detailed treatment of more or less trivial examples before reaching the first general theorem of Lie. In fact, in the present book many illustrations of the general theory are formulated only as exercises, so that Lie's three Fundamental Theorems can be developed fully in the very first chapter. Due to a similar attitude through the whole work, the present book, though of moderate extent, leads essentially farther and deeper than its predecessors, without, however, requiring too much from the reader.

The book starts out with some elementary facts regarding total and Jacobian differential systems and their generalizations. The necessary existence and uniqueness theorems of local character are used only under the restriction of analyticity. These preparatory paragraphs are followed by an explanation of the notion of a continuous transformation group and by a rather concise

* See a joint paper by myself and Trjitzinsky, *Analytic theory of linear difference equations*, Acta Mathematica, vol. 60 (1932), and a paper by Trjitzinsky entitled *Analytic theory of linear differential equations*, in vol. 62 of the same journal.

treatment of Lie's three Fundamental Theorems. Use is made of the improvements and simplifications due to Maurer, F. Schur, and the author himself. The notion of the infinitesimal transformation is introduced, as indicated by the general theory, when applied to a one-parametric group and is then adapted to an r-parametric group by imbedding a one-parametric group into the latter. The parameters belonging to a so-called canonical representation of a group are directly defined as normal ("geodesic") coordinates determined by the symmetrical components of the linear connection of a non-Riemannian geometry associated with every given transformation group. The main properties of the abelian transformation groups are deduced from the general theory.

After the development of the three Fundamental Theorems, several properties of a transformation group containing a finite number of parameters are analyzed. It is pointed out that in the case of more than one parameter every one-parametric sub-group is contained in a two-parametric one. Criteria are then given for functions which are invariants of a given group, and for the invariant varieties of the latter, especially for minimum invariant varieties, finally for sub-groups characterized by the invariance of a given variety. These considerations lead to the notion of transitivity, fundamental in the equivalence problem, and yield, in terms of the rank of an associated matrix, a test for the intransitivity of a group. A finer analysis of a given group is made possible by the notion of a system of imprimitivity.

A slight modification of the main theorem on systems of imprimitivity provides the bridge to the theory of ordinary differential equations in Lie's formulation or to the associated partial differential equations, which thus appear as illustrations of the general group theory. These topics, together with some fundamental facts on differential invariants, are extensively presented in such a way that the reader need not know their standard treatment.

A special chapter deals with the isomorphisms of continuous transformation groups and with related topics, as invariant sub-groups and factor groups and also series of composition, with application to integrable groups. Here it is advisable, although not absolutely necessary, that the reader be in control of the corresponding algebraic facts regarding finite groups.

Particular attention is devoted to the so-called structure problem of continuous groups. For the reader's convenience, the necessary algebraic tools regarding linear substitutions are not presupposed but proved and formulated in the form necessary for the application to the adjoint group of a given continuous transformation group. There follows an elegant presentation of the classical results of Lie and Killing regarding the adjoint group, which for the first time allows a deeper insight into the problem of invariant sub-groups and of simple groups on the one hand, and into the problem of integrability of a group on the other hand. The latter problem is treated with the improvements due to the thesis of E. Cartan. This chapter, leading to the investigations of Weyl, ends with a short introduction into the problems solved by the theory of semi-simple groups and with a survey of the problem of their classification. The theory of representations of "closed" continuous groups by means of linear substitutions, as recently developed by I. Schur, Weyl, and Peter, uses tools essentially distinct from those of Lie and his followers. Thus, and also since the theory of Burnside, Frobenius, and I. Schur regarding finite groups had no place in this treatise on continuous groups, the representation theory of closed groups would certainly have made the book too extensive and heterogeneous. Obviously for similar reasons, it is always presupposed that every abstract group under consideration admits of a representation such that the elements depend upon the parameters of this representation in an analytic way.

The general theory is applied to several celebrated problems of infinitesimal geometry. The first of these applications is the interrelation between simply transitive transformation groups on the one hand and non-Riemannian geometries, especially in the case of flat spaces, on the other hand, as inaugurated in 1925 by the author. This section makes possible a concise presentation of the main results of Cartan and Schouten on the geometry of a group-space and on (+), (−), and (0) parallelism, originated by the author's works. Geometric problems of a somewhat different nature, namely, the Raumproblem and related topics, are treated along the lines of the equations of Killing whose work is, however, greatly simplified and completed. The investigations of Fubini regarding intransitive and complete groups of motions and the results on surfaces admitting a deformation are developed within the framework of the general group theory of spaces. The author's investigations on motions in a linearly connected manifold also are presented. The elegant treatment of many classical problems in this and in the last chapter makes the book very useful.

The last chapter is devoted to a quite complete theory of contact transformations, developed from the standpoint of the general theory of continuous transformation groups. Criteria are developed for the different kinds of contact transformations, as homogeneous and inhomogeneous, restricted and unrestricted, infinitesimal and point transformations in contact. Particular attention is paid to the geometric or dynamical meaning of these groups. The investigations of Fine on homogeneous contact transformations of maximum rank also are included. The Hamilton-Jacobi theory is built up purely on the basis of the theory of transformation groups, that is, without resort to the calculus of variations. The author has found space to treat in a detailed and satisfactory form the fundamental connection between trajectories and waves in the Hamilton-Jacobi theory. Lie's theory of the function groups also is presented.

The author gives at the end of every section many well chosen exercises. Some of them may be solved even by a superficial reader. As to the solution of the relatively more difficult problems, exact references are given. An extensive historical list of books and papers referred to in the text is collected at the end of the book. The index is quite complete and useful.

The above account of the topics comprised in the book is incomplete and does not emphasize several original contributions of the author, but it perhaps makes clear how rich the book is both in the classical and recent aspects of the general theory and also in its applications to differential geometry, differential equations, and dynamics.

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