ALGEBRAIC DIFFERENCE EQUATIONS*

BY J. F. RITT

1. *Introduction.* It was observed by J. C. Charles in 1788† that non-linear algebraic difference equations of the first order, even if algebraically irreducible, might have more than one solution involving an arbitrary periodic function. For instance,

$$y = (x + c)^2, \tag{1}$$

where c is an arbitrary function of period unity, is a solution of

$$[y(x + 1) - y(x)]^2 - 2[y(x + 1) + y(x)] + 1 = 0. \tag{2}$$

The first member of (2) is irreducible, in the domain of rationality of all constants, as a polynomial in $y(x+1)$ and $y(x)$. Still (2) admits, in additon to (1), the solution

$$y = (ce^{\pi ix} + \frac{1}{2})^2, \tag{3}$$

where c is a function of period unity. The complete solution of (2) is given by (1) and (3).

Although the formal aspects of non-linear algebraic difference equations intrigued considerably several of the early French analysts, the general theory of such equations appears to have received, as yet, but scant attention. The first move towards a comprehensive theory seems to be contained in a recent paper by J. L. Doob and myself,‡ in which a definition was given of *irreducible system* of algebraic difference equations, and in which it was shown that every system of such equations is equivalent to a finite set of irreducible systems. The problem now is one of studying irreducible systems. Until an adequate existence theory is developed for systems of non-linear differ-

* Presented to the Society, March 31, 1934.

† See Biot, Journal de l'Ecole Polytechnique, vol. 4, cahier 11 (1802), p. 182; Poisson, ibid., p. 173; Boole, *Calculus of Finite Differences*, 3d edition, (1880), Chapter 10.

‡ American Journal of Mathematics, vol. 55 (1933), p. 505. (Designated below by α.)

ence equations,* progress in the study of irreducible systems will necessarily be mainly of an algebraic nature.

In this note, I shall study an algebraic difference equation

$$(4) \qquad A[y(x+1), y(x), x] = 0,$$

where A is a non-factorable polynomial in $y(x)$ and $y(x+1)$, with coefficients which are analytic functions of x. I shall derive a result which is roughly to the effect that if A is of degree p in $y(x)$ and of degree r in $y(x+1)$, the number of solutions of (4) which involve arbitrary functions cannot exceed the smaller of p and r.

The formulation and proof of this result will be based on the above mentioned paper of Doob and myself, and on the fourth chapter of my book *Differential Equations from the Algebraic Standpoint.*†

2. *The Theorem.* We use the terminology of α. Let A be a difference form in y, of the first order, with coefficients in a field $\mathcal{J}$. We assume that A is algebraically irreducible, that is, that A is not the product of two forms of class unity.

It is conceivable that, in the decomposition of A into irreducible systems none of which holds any other,‡ there is a system which is held by a form of zero order, that is, a polynomial in $y(x)$. Throughout our work, an irreducible system which is not held by any form of order zero will be called a *c-system.*

We shall now prove the following theorem.

THEOREM. *If A is of degree r in $y(x+1)$, there can be at most r c-systems in the decomposition of A into essential irreducible systems.*

3. *A Generalization.* To facilitate proving this theorem, we replace it by a more general result. Let Σ be a prime system of simple forms in the unknowns $y_0, y_1, \cdots, y_n$, the domain of ra-

* For references to special existence theorems of Horn and others, see Nörlund, Encyklopädie der Mathematischen Wissenschaften, vol. II C7.

† Colloquium Publications of the American Mathematical Society, vol. 14. (Designated below by β.)

‡ Such a decomposition will be called a decomposition into essential irreducible systems.

tionality being $\mathcal{J}$ (as in §2).* Let y_0 constitute a set of unconditioned unknowns for Σ. Let a basic set for Σ be

$$(5) \qquad B_1, \cdots, B_n,$$

where B_i introduces y_i.†

In what follows, we shall frequently consider, together with a simple form C, the difference form into which C goes when each y_i is understood to represent $y(x+i)$. The resulting difference form will be represented by C'. Thus (5), considered as a set of difference forms, will be represented by

$$(6) \qquad B_1', \cdots, B_n'.$$

In the same way, if Λ is any system of simple forms, the system of difference forms obtained as above from Λ will be denoted by Λ'.

We shall prove that, *if B_n is of degree r in y_n, then, in the decomposition of* (6) *into irreducible systems, there are at most r c-systems.*

4. *The Form C.* Without loss of generality, we assume that at least one c-system is present in the decomposition of (6).

Let Φ represent (5). We shall need the fact that there exists a non-zero simple form C in y_0 alone such that every solution of Φ which annuls the initial of any one of the B_i is a solution of C. In particular, every solution of Φ which is not a solution of Σ will be a solution of C. If $n=1$ in (5), we use for C the initial of B_1. We assume the truth of the result for $n=m$. What we have to prove, in the case of $n=m+1$, is the existence of a form C in y_0 alone which is annulled by every solution of

$$(7) \qquad B_1, \cdots, B_{m+1}$$

which annuls the initial I of B_{m+1}, but not the initial of any B_i with $i<m+1$.‡

* β, Chapter 4.

† By a solution of a system of simple forms in $y_0, \cdots, y_n$, we shall understand a set of functions $y_0, \cdots, y_n$, analytic in some area, which annul the forms of the system. This definition, used in β, does not conflict with the definition of solution of a set of difference forms, in which the solutions are understood to be analytic along curves going to ∞.

‡ Solutions common to (7) and an initial of B_i with $i<m+1$ are taken care of by $n=1, \cdots, m$.

Such solutions of (7) are common to I and to Λ, the prime system in $y_0, \cdots, y_m$ with $B_1, \cdots, B_m$ for basic set. As I does not hold Λ, $\Lambda+1$ is held by a non-zero form in y_0 alone.*

5. *An Adjunction.* It follows from §4 that Σ' holds all c-systems held by (6).

We return to Φ. Let y_{n+1} be a new unknown. Let D represent the simple form obtained from B_n on replacing x by $x+1$ and y_i by y_{i+1}, $(i=0, \cdots, n)$.

Let B_{n+1} be the remainder of D with respect to (5). We shall prove that B_{n+1} is of degree r in y_{n+1}.

Let H be the coefficient of $y_n{}^r$ in B_n; K that of $y_{n+1}{}^r$ in D; L that of $y_{n+1}{}^r$ in B_{n+1}. Then L is obtained by multiplying K by a product of powers of the initials in (5) and subtracting from the result a linear combination of $B_1, \cdots, B_n$. Let $L=0$. Then K holds Σ. It follows that K' holds Σ'. Then H', of which K' is a transform, holds Σ'. We shall prove that H holds Σ. If it did not, $\Sigma+H$ would be held by a non-zero simple form in y_0 alone. Then $\Sigma'+H'$ would be held by a non-zero form of order zero. As H' holds Σ', this would imply that Σ' cannot hold a c-system. Thus H holds Σ. Now this conflicts with the fact that H is reduced with respect to $B_1, \cdots, B_{n-1}$. Hence $L \neq 0$.

6. *A Special Case.* The system

$$B_1', \cdots, B_n', B_{n+1}' \tag{8}$$

is equivalent to (6). If

$$B_1, \cdots, B_n, B_{n+1} \tag{9}$$

is a basic set of a prime system, we can repeat the procedure of §5. Let us assume that this process is capable of indefinite repetition, that is, that we are led to an infinite system of simple forms

$$B_1, \cdots, B_m, \cdots \tag{10}$$

with every $B_1, \cdots, B_m$ a basic set for a prime system in

* This amounts to the fact that an irreducible algebraic manifold is of higher dimensionality than any of its proper submanifolds. It is also a simple consequence of arguments used frequently in β.

$y_0, \cdots, y_m$. We shall show that, in that case, (6) yields just one c-system.

Of course,

$$B_1', \cdots, B_m', \cdots$$

has the same manifold as (6).

Let A' be a difference form which holds one of the c-systems in the decomposition of (6). Let A be the simple form associated with A'. Let A' be of order m. The ascending set of simple forms

$$B_1, \cdots, B_m \tag{11}$$

is a basic set of a prime system Σ_m. Now A must hold Σ_m. Otherwise its solutions in common with Σ_m would annul a form in y_0 alone. The same would be true of the common solutions of A and (11) (§4). This would mean that A' holds no c-system in the decomposition of (6). Then the solutions of (11) which do not annul A are solutions of a simple form in y_0 alone. It follows that A' holds every c-system in the decomposition of (6). The fact that every form which holds a single c-system in the decomposition of (6) holds every c-system in the decomposition shows that there is only one c-system in the decomposition.

If $r=1$, the procedure of §5 can certainly be repeated indefinitely. Hence the result stated in §3 is true for $r=1$.

7. *Completion of Proof.* Let the result hold for $r=1, \cdots, s-1$. We assume B_n in (5) to be of degree s in y_n. We have only to consider the case in which the process of §5, repeated a sufficient number of times, leads to a set $B_1, \cdots, B_m$ which is not a basic set of a prime system. It is legitimate to assume that $m=n+1$.

Then, if I_i represents the initial of B_i, there exists an identity

$$I_1^{\mu_1} \cdots I_n^{\mu_n}(TB_{n+1} - G_1G_2 \cdots G_t) - K_1B_1 - \cdots - K_nB_n = 0, \tag{12}$$

where $G_1, \cdots, G_t$ are non-zero simple forms in $y_0, \cdots, y_{n+1}$, which are reduced with respect to $B_1, \cdots, B_n$. The degrees of $G_1, \cdots, G_t$ in y_{n+1} are positive, and their sum is s. Every set

$$B_1, \cdots, B_n, G_i \tag{13}$$

is a basic set of a prime system. Furthermore, T is a form in $y_0, \cdots, y_n$ which does not hold Σ of §3.*

By (12), every solution of

$$B_1, \cdots, B_{n+1} \tag{14}$$

which does not annul $I_1 \cdots I_n$ is a solution of some system (13). This shows that every c-system in the decomposition of

$$B_1', \cdots, B_{n+1}' \tag{15}$$

is held by some system

$$B_1', \cdots, B_n', G_i'. \tag{16}$$

On the other hand, because T does not hold Σ, (12) shows that every c-system in the decomposition of any system (16) is held by (15).

Thus we obtain the c-systems in the decomposition of (15) by collecting the c-systems in the decompositions of the t sets (16) and suppressing certain of the systems which are held by others. As each G_i is of degree less than s in y_{n+1}, our result holds for each system (16). Thus we cannot get more than s c-systems from (15). This completes the proof.

8. *An Extension.* Suppose that the open region in which the functions in $\mathfrak{J}$ are meromorphic contains $x-1$, as well as $x+1$, when it contains x. Suppose also that $\mathfrak{J}$ contains, together with any function $f(x)$, the function $f(x-1)$ as well as $f(x+1)$. Finally, let us understand solutions of forms to be made up of functions analytic along a curve which contains $x-1$, as well as $x+1$, when it contains x. A few simple considerations show that, under these circumstances, the number of c-systems in the decomposition of A of §2 cannot exceed the degree of A in $y(x)$.

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* β, §47. The use of more than two G_i requires no essential changes in the discussion of β.