nacci series 0, 1, 1, 2, 3, 5, 8, 13, $\cdots$ giving the values of the Lucas function U_n associated with the polynomial x^2-x-1. This polynomial is irreducible modulo 13, so that the period of the Fibonacci series modulo 13 gives the period of the mark α associated with x^2-x-1 in the finite field of order 13^2. We have $\omega=7$, norm $\alpha=-1$, $\theta=2$, $k=2$, $\sigma=2$, $p-1=12$. Hence (2) becomes $(2, 2)\ |\delta|\ (2, 12)$, so that $\delta=2$. Hence the period is 28, which is easily verified directly. It seems quite difficult to determine the exact value of δ in all cases.*

CALIFORNIA INSTITUTE OF TECHNOLOGY

ON A PROBLEM OF KNASTER AND ZARANKIEWICZ†

BY J. H. ROBERTS

Knaster and Zarankiewicz have proposed the following problem:‡ "Does every continuum A contain a subcontinuum B such that $A-B$ is connected?" Knaster has shown,§ by an example in 3-space, that the answer is in the negative. In the present paper an example is given of a *plane* continuum M such that every non-degenerate proper subcontinuum of M disconnects M.

The point sets considered in this paper all lie in a plane.

DEFINITION OF $F(C; X, Y; \epsilon)$. Let C be any simple closed curve, X and Y distinct points of C, and ϵ any positive number. There exists a finite set of points $A_1, A_2, \cdots, A_n$, $(n>2)$, such that (a) $A_1+A_2+\cdots+A_n$ contains $X+Y$, (b) $A_1, A_2, \cdots, A_n$ lie on C in the order $A_1A_2\cdots A_nA_1$, and (c) A_i and A_{i+1} (subscripts are to be reduced modulo n) are the end points of an arc t_i of diameter $<\epsilon$ which is a subset of C not containing A_{i+2}. There exists a set of mutually exclusive arc segments $v_1, v_2, \cdots, v_n$ lying within C such that v_i+t_i is a simple closed curve w_i of diameter $<\epsilon$. Let J denote the simple closed curve

---

* See the discussion at the close of my paper, Transactions of this Society, vol. 33 (1931), p. 165.

† Presented to the Society, December 1, 1933.

‡ Fundamenta Mathematicae, vol. 8 (1926), Problem 42, p. 376.

§ B. Knaster, *Sur un continu que tout sous-continu divise*, Proceedings of the Polish Mathematical Congress, 1929, p. 59.

$\sum_1^n A_i + v_i$. There exist n infinite sequences of simple closed curves C_{ij}, $(i=1, 2, \cdots, n; j=1, 2, \cdots)$, such that (1) C_{ij} contains A_i but otherwise lies within J, (2) the sequence $C_{i1}, C_{i2}, C_{i3}, \cdots$ has as sequential limit set the arc $A_i + v_i + A_{i+1}$, (3) C_{ij} is of diameter $<\epsilon$, (4) $C_{ij} \cdot C_{ik} = A_i$, $(j \neq k)$, and $C_{ij} \cdot C_{hk} = 0$, $(i \neq h)$, and (5) no point of C_{ij} lies within any C_{hk}. The set $F(C; X, Y; \epsilon)$ is defined as the sum of all the curves C_{ij} and the n curves w_i:

$$F(C; X, Y; \epsilon) = \sum_{i=1}^{n} \left[w_i + \sum_{j=1}^{\infty} C_{ij} \right].$$

DEFINITION OF M. Let E be any simple closed curve, X and Y any two points of E. Let K_1 denote a set $F(E; X, Y; 1)$. Then $K_1 = \sum_{i=1}^{\infty} E_{1i}$, where for each i, E_{1i} is a simple closed curve of diameter <1, and the common part of E_{1i} and the sum of the other curves $E_{11}, E_{12}, \cdots$ either is one point, or is two points. Thus E_{1i} contains distinct points X_{1i} and Y_{1i} such that no other point of E_{1i} belongs to $E_{1j}, (i \neq j)$. For each i let G_{1i} be a set $F(E_{1i}; X_{1i}, Y_{1i}; 1/2)$ and let K_2 be $G_{11} + G_{12} + \cdots$.

Suppose $K_1, K_2, \cdots, K_n$, $(n>1)$, have been defined, K_1 being as defined above and, for each i, the following properties obtain:

I. K_i is the sum of a countable set of simple closed curves $E_{i1}, E_{i2}, \cdots$.

II. Each curve E_{ih} has, in common with the sum of the other curves $E_{i1}, E_{i2}, \cdots$, either one point or two points.

III. X_{ih} and Y_{ih} are distinct points of E_{ih} such that no other point of E_{ih} belongs to the sum of the other curves $E_{i1}, E_{i2}, \cdots$.

IV. No point is common to the interiors of two curves E_{ih} and E_{ik}, $(h \neq k)$.

V. K_{i+1}, $(i<n)$, is a subset of the sum of K_i and the interiors of all the curves $E_{i1}, E_{i2}, \cdots$.

VI. The subset of K_{i+1}, $(i<n)$, which lies on and within E_{ih} is a set $F(E_{ih}; X_{ih}, Y_{ih}; 1/[i+1])$.

For $n=2$, the sets K_1 and K_2 defined above have these properties. For each $i, (i \leqq n)$, let U_i be the set of all points of K_i, each of which belongs to at least two curves of the set $E_{i1}, E_{i2}, \cdots$, and let D_i denote K_i plus the interiors of all the curves E_{i1}, $E_{i2}, \cdots$.

For each k let G_{nk} be a set $F(E_{nk};\ X_{nk},\ Y_{nk};\ 1/[n+1])$, and let K_{n+1} be $G_{n1}+G_{n2}+\cdots$. Then it readily follows that the sequence $K_1, K_2, \cdots, K_n, K_{n+1}$ has the properties I–VI above. Hence there is an infinite sequence $K_1, K_2, \cdots$ with properties I–VI, K_1 being a set $F(E;\ X,\ Y;\ 1)$. Let M be $K_1+K_2+\cdots$ plus all limit points. This is the same as the common part of $D_1, D_2, \cdots$.

PROOF THAT $M-H$ IS NOT CONNECTED. Suppose H is a nondegenerate proper subcontinuum of M. Suppose $M-H$ is connected. Now the components of $M-U_n$ are of diameter $<1/n$. Hence there exists an n such that H contains a point P of U_n. It will be shown that if H contains a point of U_n, then it contains all of U_n. In view of this, and the fact that U_n is a subset of U_{n+1} and that $M=(U_1+U_2+\cdots)$ plus limit points, it follows that $H=M$, which is a contradiction.

It remains to show that if H contains a point P of U_n, then it contains all of U_n. Let h be such that P belongs to E_{nh}. The subset of K_{n+1} which lies on and within E_{nh} is a set $F(E_{nh};\ X_{nh},\ Y_{nh};\ 1/[n+1])$. The points of U_{n+1} in this set can be labeled $B_1, B_2, \cdots, B_k$, so that they lie on E_{nh} in the order $B_1B_2\cdots B_kB_1$. Now each of the infinity of components of $K_{n+1}-B_i$ is a subset of a different component of $M-B_i$. Hence if H contains B_i, and $M-H$ is connected, H must contain all save one of these components. But B_{i+1} is a limit point of the sum of the components of $K_{n+1}-B_i$. Hence, if H contains B_i, it contains B_{i+1}. But for some i, $P=B_i$. Thus H contains all the points of U_{n+1} on E_{nh}, and therefore the one or two points of U_n on E_{nh}. Now any two curves E_{nh} and E_{nk}, of the set $E_{n1}, E_{n2}, \cdots$, can be joined by a finite chain $L_1, L_2, \cdots, L_e$ of curves of the set $E_{n1}, E_{n2}, \cdots$, L_1 having a point in common with E_{nh}, L_i having a point in common with L_{i+1}, $(i<e)$, and L_e having a point in common with E_{nk}. Since these common points are in U_n, and H contains a point of U_n in E_{nh}, it readily follows, by repeated application of the above argument, that H contains every point of U_n in $E_{nh}+L_1+L_2+\cdots+L_e+E_{nk}$, and therefore H contains every point of U_n.

DUKE UNIVERSITY