LINEAR ASSOCIATIVE ALGEBRAS OF INFINITE ORDER WHOSE ELEMENTS SATISFY FINITE ALGEBRAIC EQUATIONS*

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1. *Introduction.* It is the purpose of this paper to investigate linear associative algebras of infinite order, whose elements satisfy finite algebraic equations with coefficients in a field Ξ. The definition† of an algebra, and the first three postulates‡ assumed, will be the same as those employed by L. E. Dickson for a finite algebra, but in place of Dickson's postulates for a finite basis we shall employ postulates IV and V as follows.

POSTULATE IV. *There exists in A a set of elements§ U of such a nature that for every $a \neq 0$ there is determined uniquely a positive integer n, a set of distinct elements $u_1, u_2, \cdots, u_n$ of U, and n non-zero elements $\xi_1, \xi_2, \cdots, \xi_n$ of Ξ such that $a = \sum_{i=1}^{n} \xi_i u_i$.*

POSTULATE V. *For every element a of A there exists a polynomial function $h(\lambda)$, with coefficients in Ξ, such that $h(a) = 0$.*

2. *Idempotent Elements; Semi-Nilpotent Algebras.*

THEOREM 1. *Every algebra A contains an idempotent element unless all its elements are nilpotent.*

For if a is any non-zero non-nilpotent element of A whose minimum‖ equation is $g(\lambda) = 0$, of degree n, then the finite subalgebra $B = (a, a^2, a^3, \cdots, a^n)$ of A, contains an idempotent element.

THEOREM 2. *If an algebra A is not semi-nilpotent,¶ but con-*

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† L. E. Dickson, *Algebras and their Arithmetics*, University of Chicago Press, pp. 9–11, cited hereafter as Dickson.

‡ For the convenience of the reader, references will be made to Dickson, wherever possible, whether or not this constitutes the original source.

§ The set U is not assumed enumerable except in the example of §2, the subalgebra M of A in Theorem 11, and in Theorem 13.

‖ The equation $g(\lambda) = 0$ of lowest degree, with rational coefficients and leading coefficient unity, for which $g(a) = 0$, will be called the minimum equation of a.

¶ An algebra A will be called semi-nilpotent if all its elements are nilpotent, and semi-simple if it contains no properly nilpotent elements.

tains a maximal semi-nilpotent invariant sub-algebra K, then K is unique.*

For if K' is any other semi-nilpotent invariant sub-algebra of A, then $K+K'$ is invariant in A. Moreover, if k_1 and k_1' are any two elements of K and K', respectively, then $(k_1+k_1')^{\alpha\beta} = (k_\alpha+k_1'^{\alpha})^\beta = k_\alpha^\beta = 0$, where k_α is in K and the indices† of k_1' and k_α are α and β, respectively. Therefore $K+K'$ is a semi-nilpotent invariant sub-algebra of A. But K is maximal, hence $K+K'=K$, or K' is contained in K.

LEMMA. *Any non-zero element a of an algebra A, which is not properly nilpotent in A, possesses an inverse with respect to some idempotent element e of A, that is, an element x such that $ax=e$.*

For otherwise, the sub-algebra aA contains no idempotent element and hence is semi-nilpotent and a is properly nilpotent in A, contrary to hypothesis.

THEOREM 3. *If a and b are any two properly nilpotent elements of an algebra A, then their sum $a+b$ is also properly nilpotent in A.*

The contrary assumption necessitates the existence in A of elements x and e such that $(a+b)x = ax+bx = p+q = e$, where e is idempotent and epe and eqe are zero or properly nilpotent. Therefore, if α is the index of epe, $(epe)^\alpha = (e-eqe)^\alpha = 0$, and hence $e = a_1qe$, (a_1 in A). But a_1qe is properly nilpotent in A, whereas e is idempotent. This contradiction implies that $a+b$ is properly nilpotent in A.

THEOREM 4. *An algebra A which is not semi-nilpotent contains properly nilpotent elements if and only if it possesses a maximal semi-nilpotent invariant sub-algebra K, and then the properly nilpotent elements of A coincide with the non-zero elements of K.*

The first part of the theorem follows at once from the definition of K. To prove the second part, represent the aggregate of properly nilpotent elements in A by B. By means of Theorem 3 we can show that B is a semi-nilpotent invariant sub-algebra of A, and since it contains all the properly nilpotent elements of A it is maximal and hence is identical with K (Theorem 2).

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* Dickson, p. 32.

† If α is an integer such that $a^\alpha = 0$, but $a^{\alpha-1} \neq 0$, α is the index of a.

An algebra A may be decomposed into the form

$$A = I + eR + Le + eAe, \tag{1}$$

in which e is idempotent, I contains all elements x of A such that $ex = xe = 0$, $R = I + eR$ contains all elements y of A such that $ye = 0$, and $L = I + Le$ contains all elements z of A such that $ez = 0$.

THEOREM 5. *If e is a principal idempotent element of A, every non-zero element of I, L, and R in* (1) *is properly nilpotent.*

PROOF. $(Le)^2 = 0 = (eR)^2$. Since $eLR = LRe = 0$, it follows that $LR \leqq I$. If x and z are any two non-zero elements of I and Le, respectively, then $(x+z)^n = x^n + z_{n-1}$, (z_{n-1} in Le). Since x is nilpotent with an index α, $(x+z)^{2\alpha} = z_{\alpha-1}^2 = 0$, ($z_{\alpha-1}$ in Le). Thus each element of L is zero or nilpotent. Likewise each element of R is zero or nilpotent.

$ARe = 0 = Re$, and $eLA = 0 = eL$, hence $AR \leqq R$ and $LA \leqq L$. Therefore the elements of R and L, and hence of their intersection I, are properly nilpotent in A.

THEOREM 6. *Every algebra A with a principal idempotent element, but no principal unit, has a semi-nilpotent invariant sub-algebra K.*

This theorem is a direct consequence of Theorems 4 and 5.

*Any semi-simple algebra A of finite order which is not simple is reducible.** The proof of this theorem depends upon the theorem, *Any invariant sub-algebra of a semi-simple algebra of finite order possesses a principal unit.* The following example exhibits the failure of the latter theorem for infinite algebras, and the sub-algebra B illustrates the failure of two other theorems.

(1) *Every finite linear associative algebra which is not nilpotent contains a principal idempotent element.**

(2) *Every finite algebra with no principal unit has a nilpotent invariant sub-algebra.**

EXAMPLE. Let $A = (e, u_1, u_2, u_3, \cdots)$ in which $eu_i = u_ie = u_i$, $e^2 = e$, $u_i^2 = u_i$, $u_iu_j = u_ju_i = 0$, $i \neq j$. Then A is a semi-simple algebra with the principal unit e, and an invariant sub-algebra

* Dickson, pp. 49–53.

$B=(u_1, u_2, u_3, \cdots)$ which contains no principal unit, no principal idempotent element and no nilpotent invariant sub-algebra.

3. *Difference Algebras.** THEOREM 7. *If an algebra A is not semi-nilpotent, but contains the maximal semi-nilpotent invariant sub-algebra K, then the difference algebra $A-K$ contains no semi-nilpotent invariant sub-algebra.*

For suppose $A-K$ has a semi-nilpotent invariant sub-algebra K_1. The elements of $A-K$ are classes $[a]$ of elements of A. Represent by B all elements of A belonging to classes $[b]$ of $A-K$ which are in K_1. From the definition of K_1, B is an algebra. If a and b are any two elements of A and B, respectively, then $[a][b]=[ab]$ is in K_1 and also in $B-K$. Hence B is invariant in A, and $B-K=K_1$, or $B>K$. All elements of B are nilpotent and hence B is a semi-nilpotent invariant sub-algebra of A. But $B>K$ contrary to the hypothesis on K, therefore $A-K$ contains no semi-nilpotent invariant sub-algebra.

THEOREM 8. *Every idempotent class $[u]$ of $A-K$ contains idempotent elements of A.*

PROOF. $[u]=[u^2]=[u^3]=\cdots=[u^\alpha]$, and $[u]\neq[0]$. Hence u is not nilpotent. If the minimum equation of u is of degree n, the finite sub-algebra $B=(u, u^2, \cdots, u^n)$, of A, contains an idempotent element $e=\xi_1 u+\xi_2 u^2+\cdots+\xi_n u^n$, ($\xi_i$ in Ξ), (Theorem 1). Hence $[e]=\xi[u]$, ($\xi=\xi_1+\xi_2+\cdots+\xi_n$). Therefore, $\xi[u]=[e]=[e]^2=\xi^2[u]^2=\xi^2[u]$, and hence $\xi=1$, since $\xi=0$ implies $[e]=[0]$ and e nilpotent. Thus e is an idempotent element of A, in class $[u]$ of $A-K$.

THEOREM 9. *If u is a primitive* idempotent element of A, and K is a maximal semi-nilpotent invariant sub-algebra of A, then $[u]$ is a primitive idempotent element of $A-K$.*

THEOREM 10. *If e is a principal* idempotent element of A, then $[e]$ is a principal idempotent element of $A-K$ and is identical with its principal unit.**

Theorem 9 is easily proved by means of Theorems 1 and 8, while Theorem 10 follows readily from Theorems 5 and 6.

* Dickson, pp. 36–40, 55, 49, 15.

THEOREM 11. *If A has the maximal semi-nilpotent invariant sub-algebra K, and $A-K$ contains a simple* matric algebra M, with an enumerable base, then A contains a sub-algebra equivalent to M.*

PROOF. Let $M=[\epsilon_{ij}]$ be a simple matric algebra of classes, in which $[\epsilon_{ij}][\epsilon_{jk}]=[\epsilon_{ik}]$, $[\epsilon_{ij}][\epsilon_{hk}]=[0]$, $(j\neq h,\ i,\ k,\ j,\ h=1,\ 2,\ 3,\ \cdots)$. With e_{11} in $[\epsilon_{11}]$, (Theorem 8), as a basis for induction we shall first prove that A contains idempotent elements $e_{11}, e_{22}, \cdots$ whose products in pairs are zero and such that e_{ii} is in $[\epsilon_{ii}]$.

Let $s=\sum_{i=1}^{n-1}e_{ii}$, where $e_{ii}e_{jj}=0$, $i\neq j$, and e_{ii} is in $[\epsilon_{ii}]$. Let b_n be any element of A in $[\epsilon_{nn}]$ and let $a_n=b_n-sb_n-b_ns+sb_ns$. Then $e_{ii}a_n=0=a_ne_{ii}$, and $[a_n]=[b_n]=[\epsilon_{nn}]$, and hence $[a_n][\epsilon_{ii}]=[0]=[\epsilon_{ii}][a_n]$, $(i=1, 2, \cdots, n-1)$. The sub-algebra $B=(a_n, a_n^2, \cdots, a_n^k)$, of A, contains an idempotent element e_{nn} in $[\epsilon_{nn}]$, (Theorem 8), such that $e_{nn}e_{ii}=0=e_{ii}e_{nn}$, $i\neq n$. Therefore A contains idempotent elements $e_{11}, e_{22}, e_{33}, \cdots$ whose products in pairs are zero and such that e_{ii} is in $[\epsilon_{ii}]$ of M.

Now consider the non-zero elements a_{1j} and b_{j1} of A, in $[\epsilon_{1j}]$ and $[\epsilon_{j1}]$ of M, respectively. Let $e_{1j}=e_{11}a_{1j}e_{jj}$, and $a_{j1}=e_{jj}b_{j1}e_{11}$. Then $[e_{1j}]\ [a_{j1}]=[\epsilon_{11}]$ and hence $e_{1j}a_{j1}=e_{11}+k$, (k in K, with index α). Moreover, $e_{11}\epsilon_{1j}a_{j1}=e_{1j}a_{j1}$, and hence $e_{11}(e_{11}+k)=e_{11}+k$, or $e_{11}k=k$. Similarly, $ke_{11}=k$. Let $e_{j1}=a_{j1}-a_{j1}k+a_jk^2-a_jk^3+\cdots+(-1)^{\alpha-1}\,a_{j1}k^{\alpha-1}$. Then

$$e_{1j}e_{j1}=(e_{11}+k)-(k+k^2)+\cdots+(-1)^{\alpha-1}(k^{\alpha-1}+k^{\alpha})$$
$$=e_{11}+k^{\alpha}=e_{11}.$$

Similarly, $e_{j1}e_{1j}=e_{jj}+k_1$, (k_1 in K). Hence from the definitions of a_{j1}, e_{1j}, and e_{j1}, it follows that $e_{jj}k_1=k_1e_{jj}=k_1$. Therefore $(e_{jj}+k_1)^2=e_{jj}+2k_1+k_1^2=(e_{j1}e_{1j})^2=e_{jj}+k_1$, and $k_1^2=-k_1$, from which $k_1^{2\alpha}=-k_1=0$, since k_1 is nilpotent. Therefore, $e_{j1}e_{1j}=e_{jj}$.

Let $e_{ij}=e_{i1}e_{1j}$, (e_{i1} in $[\epsilon_{i1}]$, e_{1j} in $[\epsilon_{1j}]$). Then e_{ij} is in $[\epsilon_{ij}]$. From the definitions of e_{ij} and e_{1j}, and preceding relations, it follows that

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* If A is an algebra of matrices having only a finite number of non-zero elements in each matrix, and if the base elements (u_k), $(k,\ h=1,\ 2,\ \cdots)$, of A are matrices whose elements are all zero except that in the kth row and hth column, which is unity, then the algebra $M=(\epsilon_{ij})$ equivalent to A, in which (i, j) have the same range as (k, h), will be called a simple matric algebra with the base elements ϵ_{ij}. Thus $\epsilon_{ij}\epsilon_{jk}=\epsilon_{ik}$, $\epsilon_{ij}\epsilon_{hk}=0$, $h\neq j$.

$$e_{ij}e_{jk} = e_{ik}, \quad \text{and} \quad e_{ij}e_{hk} = (e_{i1}e_{1j}e_{jj})(e_{hh}e_{h1}e_{1k}) = 0, \quad j \neq h.$$

This completes the proof that A contains elements e_{ij} in classes $[\epsilon_{ij}]$ of M, respectively, and such that the elements e_{ij} constitute the base of a sub-algebra of A which is equivalent to M.

4. *Canonical Form of the Matrix R.* Every linear associative algebra satisfying the postulates of this paper is equivalent* to a matric algebra, and hence an element of the algebra, and its matric representation, must satisfy the same minimum equation. Every finite matrix can be reduced to a rational canonical† form, but this is not true for matrices of infinite order.

Let $R_a = (\rho_{kj})$ correspond to $a = \sum_i \xi_i u_i$, where $\rho_{kj} = \sum_i \xi_i \gamma_{ijk}$, and γ_{ijk} is the coefficient of u_k in the product $u_i u_j$. Since the number of terms u_k in any product $u_i u_j$ is finite, R_a will have but a finite number of non-zero elements in any column. Let $W_i = \sum_k \rho_{ki} u_i$, where k has an infinite range, but only a finite number of the ρ_{ki} are not zero. Let $x_{11} = \sum_i c_i u_i$, and $X_{11} = \sum_i c_i W_i$, be any two corresponding finite linear functions of u_i and W_i, respectively, c_i in Ξ. Then $X_{11} = \sum_{ik} c_i \rho_{ki} u_k = x_{12}$, and similarly, $X_{12} = \sum_{ik} c_i \rho_{ki} W_k = \sum_{ikh} c_i \rho_{ki} \rho_{hk} u_h = \sum_{ih} c_i \rho'_{hi} u_h = x_{13}$, $X_{13} = \sum_{ih} c_i \rho'_{hi} W_h = \sum_{ij} c_i \rho^{(2)}_{ji} u_j = x_{14}$, etc., where $\rho^{(r-1)}_{kj}$ is the element in the kth row and jth column of $(\rho_{kj})^r$. These relations may be written more briefly in the form

$$(1) \quad \begin{gathered} x_{11} = \sum_i c_i u_i,\ X_{11} = \sum_{ij} c_i \rho_{ji} u_j = x_{12},\ X_{12} = \sum_{ij} c_i \rho'_{ji} u_j = x_{13}, \\ \cdots,\ X_{1r} = \sum_{ij} c_i \rho^{(r-1)}_{ji} u_j = x_{1r+1}, \cdots . \end{gathered}$$

If the minimum equation of a is $\lambda^m + \xi_m \lambda^{m-1} + \cdots + \xi_1 \lambda + \xi_0 = 0$, then

$$\begin{gathered} \rho^{(m-1)}_{ji} + \xi_m \rho^{(m-2)}_{ji} + \cdots + \xi_1 \rho_{ji} = 0,\ (i \neq j,\ i, j = 1, 2, 3, \cdots), \\ \rho^{(m-1)}_{ii} + \xi_m \rho^{(m-2)}_{ii} + \cdots + \xi_1 \rho_{ii} + \xi_0 = 0. \end{gathered}$$

Therefore

$$X_{1m} + \xi_m X_{1m-1} + \cdots + \xi_1 X_{11} + \xi_0 x_{11} = 0,$$

* M. H. Ingraham, this Bulletin, vol. 32 (1926), p. 589.

† L. E. Dickson, *Modern Algebraic Theories*, p. 89.

which proves the following theorem.

THEOREM 12. *If* (ρ_{kj}) *satisfies an equation of degree* m, *with rational coefficients, the number of linearly independent linear functions* $x_{11}, X_{11}, \cdots, X_{1i-1}$, *as determined in* (1), *is* $\alpha \leqq m$.

In what follows we shall write (1) in the form

$$(1') \qquad \begin{aligned} X_{11} &= x_{12}, \quad X_{12} = x_{13}, \cdots, \quad X_{1\alpha-1} = x_{1\alpha}, \\ X_{1\alpha} &= \sum_{i=1}^{\alpha} \xi_{1i} x_{1i} = [x_{11}, \cdots, x_{1\alpha}]. \end{aligned}$$

Any finite linear expression $\sum_i c_i u_i$ leads to a chain of the type of (1′). The maximal length of all such chains is $\alpha_1 \leqq m$.

THEOREM 13. *If* $R_a = (\rho_{kj})$ *corresponds to the element* $a = \sum_i c_i u_i$ *of a rational linear associative algebra* A, *with the enumerable base* $(u_1, u_2, u_3, \cdots)$, *then* R_a *may be reduced to a canonical form defined by*

$$(2) \qquad \begin{aligned} X_{j_i 1} &= x_{j_i 2}, \cdots, \quad X_{j_i \alpha_i - 1} = x_{j_i \alpha_i}, \\ X_{j_i \alpha_i} &= [x_{j_i 1}, x_{j_i 2}, \cdots, x_{j_i \alpha_i}], \\ &(i = 1, 2, \cdots, p \leqq \alpha_1; j_i = 1, 2, 3, \cdots), \end{aligned}$$

in which $X_{j_i 1}, X_{j_i 2}, \cdots, X_{j_i \alpha_i}$ *are the elements of the* j_i*th chain of length* α_i, *each of which is linearly independent of all preceding chains of length* $\alpha_1, \alpha_2, \cdots, \alpha_{i-1}$ *and the preceding elements of the* j_i*th chain of length* α_i. *Moreover,* α_1 *is the maximal length of all possible chains, and in general,* α_i *is the maximal length of all chains of length less than* α_{i-1} *which are linearly independent of all chains of length* $\alpha_1, \alpha_2, \cdots, \alpha_{i-1}$.

Proofs of this theorem for the finite case that have been given by L. E. Dickson (*Modern Algebraic Theories*, p. 90), or M. H. Ingraham (this Bulletin, vol. 39, p. 379), may be extended to the infinite case, provided we secure the leaders of successive chains as follows.

The leader, $x_{11} = \sum_i c_i u_i$, of the first chain of length α_1 may be obtained from successive trials of the linear functions $\sum_i c_i u_i$ such that $\sum i = n$, $(n = 1, 2, \cdots, k_1)$. The leaders of the succeeding chains of length α_1 may be obtained in the same way with $\sum i = n$, $(n = k_1, k_1+1, k_1+2, \cdots, k_2; k_2, k_2+1, \cdots, k_3;$ etc.). To obtain the chains of length α_2, α_3, etc., repeat the entire process for α_2, then for α_3, etc. With (1′) as a basis for induction we can prove that the kth chain of length α_1 can be expressed in the form

$$(3)\ X_{k1} = x_{k2}, \cdots, X_{k\alpha_1-1} = x_{k\alpha_1}, X_{k\alpha_1} = [x_{k1}, x_{k2}, \cdots, x_{k\alpha_1}].$$

Similarly, all chains of length α_i can be reduced to forms of the type of (3). We can thus reduce (ρ_{kj}) to the canonical form defined by (2), in which there are infinitely many partial transformations, but not more than α_1 different lengths to the chains.

Each of the base elements $u_1, u_2, u_3, \cdots$, either is itself an element of (2), or else is a finite linear combination of such elements. For, in the process of determining the complete canonical transformation, any u_i which is finitely linearly independent of all previously determined x_{jk} is taken as the leader of a new chain and is therefore itself an x_{jk}. Hence it is possible to determine uniquely each of the variables u_i in terms of the variables x_{jk}.

The characteristic determinant of the matrix of any chain is divisible by that of any other chain of equal or lower order. Hence the minimum function of (ρ_{kj}) is identical with that of the chain of maximal length, and if (ρ_{kj}) satisfies no finite equation it cannot be reduced to a rational canonical form with a chain of maximal length α.

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