DERIVATIVES, DIFFERENCE QUOTIENTS, AND TAYLOR'S FORMULA*

BY HASSLER WHITNEY†

1. *Introduction.* Let $f(x)$ be defined in the closed interval I. If $f(x)$ has a continuous mth derivative, it can be expanded in a Taylor's formula with $m+1$ terms plus remainder; the mth difference quotient of $f(x)$ approaches $d^m f(x)/dx^m$ uniformly. If $f(x)$ is a polynomial of degree at most $m-1$, then the mth difference quotient is identically zero. The object of the present note is to prove converses of these theorems. The results hold also in an open interval, as they hold in every closed subinterval.

2. *Difference Quotients.* Given a function $f(x)$ defined in I, the pth difference quotient is defined by the equations $\Delta_h^0 f(x) = f(x)$, and

$$\begin{aligned} \Delta_h^p f(x) &= \frac{1}{h^p}\sum_{i=0}^{p}(-1)^{p-i}\binom{p}{i}f(x+ih) \\ &= \frac{1}{h}[\Delta_h^{p-1}f(x+h) - \Delta_h^{p-1}f(x)] \end{aligned} \tag{1}$$

for $p>0$. We say $\Delta_h^p f(x) \to f_p(x)$ uniformly in I if for every $\epsilon>0$ there is a $\delta>0$ such that $|\Delta_h^p f(x) - f_p(x)| < \epsilon$ for every x in I and every h, $|h|<\delta$.‡

(a) Suppose $f_0(x), f_1(x), \cdots, f_m(x)$ are defined in I, and

$$f_0(x+h) = f_0(x) + \frac{1}{1!}f_1(x)h + \cdots + \frac{1}{m!}f_m(x)h^m + R(x,h), \tag{2}$$

where $R(x,h)/h^m \to 0$ uniformly in I as $h\to 0$. If we form the pth difference quotient, we find§

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† National Research Fellow.

‡ We always suppose that the values of x under consideration (here, $x+ih$ for $i=0,\cdots,p$) lie in I.

§ If we solve the linear equations (setting $0^0=1$)

$$0^j u_0 + 1^j u_1 + \cdots + p^j u_p = p!\delta_{jp}, \qquad (j=0,\cdots,p),$$

we find $u_i = (-1)^{p-i}\binom{p}{i}$; hence $\sum_{i=0}^{p}(-1)^{p-i}\binom{p}{i}i^j = 0$, $(j<p)$, and $=p!$, $(j=p)$.

$$(3)\quad \Delta_h^p f_0(x) = \frac{1}{h^p}\sum_{i=0}^{p}(-1)^{p-i}\binom{p}{i}\left[\sum_{j=0}^{m}\frac{f_j(x)}{j!}(ih)^j + R(x,ih)\right]$$
$$= \sum_{j=0}^{m}\frac{f_j(x)}{j!}h^{j-p}\sum_{i=0}^{p}(-1)^{p-i}\binom{p}{i}i^j + \frac{1}{h^p}R_p(x,h)$$
$$= f_p(x) + \sum_{j=p+1}^{m}\frac{f_j(x)}{j!}h^{j-p}\sum_{i=0}^{p}(-1)^{p-i}\binom{p}{i}i^j$$
$$+ \frac{1}{h^p}R_p(x,h),$$

where

$$(4)\qquad R_p(x,h) = \sum_{i=0}^{p}(-1)^{p-i}\binom{p}{i}R(x,ih).$$

From (2) we see that $f_0(x)$ is continuous; hence $\Delta_h^p f_0(x)$ is continuous (h fixed, $p=1,\cdots,m$). Setting $p=m$ in (3), we see that $\Delta_h^m f_0(x)\to f_m(x)$ uniformly; hence $f_m(x)$ is continuous. Having proved that $\Delta_h^q f_0(x)\to f_q(x)$ uniformly and $f_q(x)$ is continuous, ($q=m, m-1,\cdots,p+1$), (3) shows that $\Delta_h^p f_0(x)\to f_p(x)$ uniformly, and hence $f_p(x)$ is continuous, ($p=1,\cdots,m$).

(b) Suppose $f(x)=f_0(x)$ and $d^pf(x)/dx^p$ exists and equals the continuous function $f_p(x)$, ($p=1,\cdots,m$). Then (2) holds, by Taylor's theorem, and hence $\Delta_h^p f(x)\to d^pf(x)/dx^p$ uniformly, ($p=1,\cdots,m$).

(c) Suppose $f(x)=a_0+a_1x+\cdots+a_mx^m$ is a polynomial of degree at most m. Then a Taylor's formula (2) holds with $f_m(x)\equiv m!a_m$ and $R(x,h)\equiv 0$, and using (3) with $p=m$, we have $\Delta_h^m f(x)\equiv m!a_m$. If $f(x)$ is of degree at most $m-1$, $\Delta_h^m f(x)\equiv 0$.

3. *Rolle's Theorem for Difference Quotients.* We shall prove first the following lemma.

LEMMA 1. *If $k_0<k_1<\cdots<k_m$ are integers, h is >0, and*

$$(5)\qquad (-1)^{m-i}f(x+k_ih)\geqq 0,\qquad (i=0,\cdots,m),$$

then there is an integer $k_0^{(m)}$ ($k_0\leqq k_0^{(m)}<k_m$) such that

$$(6)\qquad \Delta_h^m f(x+k_0^{(m)}h)\geqq 0.$$

As $f(x+k_mh)\geqq 0$ and $f(x+k_{m-1}h)\leqq 0$, there is a $k_{m-1}^{(1)}$,

$(k_{m-1} \leqq k_{m-1}^{(1)} < k_m)$, such that $\Delta_h^1 f(x + k_{m-1}^{(1)} h) \geqq 0$. In this manner we find numbers $k_0^{(1)}, \cdots, k_{m-1}^{(1)}$, $(k_{i-1} \leqq k_{i-1}^{(1)} < k_i)$, such that

$$(-1)^{m-1-i} \Delta_h^1 f(x + k_i^{(1)} h) \geqq 0, \quad (i = 0, \cdots, m-1).$$

By the same method, using $\Delta_h^1 f(x)$ in place of $f(x)$, we find numbers $k_0^{(2)}, \cdots, k_{m-2}^{(2)}, (k_{i-1}^{(1)} \leqq k_{i-1}^{(2)} < k_i^{(1)})$, such that

$$(-1)^{m-2-i} \Delta_h^2 f(x + k_i^{(2)} h) \geqq 0, \quad (i = 0, \cdots, m-2).$$

Continuing in this manner we find, finally, the required number $k_0^{(m)}$.

4. THEOREM 1. *Let $f(x)$ be measurable in the closed interval I. A necessary and sufficient condition that $f(x)$ be a polynomial of degree at most $m-1$ is that $\Delta_h^m f(x) \to 0$ uniformly.**

The theorem is not true without the measurability condition. For the discontinuous functions defined by G. Hamel† are seen to have the property $\Delta_h^r f(x) \equiv 0$.

The necessity of the condition follows from §2 (c); we must prove the sufficiency. We shall prove first the following lemma.

LEMMA 2. *Let $f(x)$ satisfy the conditions of the theorem. Let $r_1, r_2, \cdots, r_m$ be rational numbers. Take a fixed number a and a fixed t. Let $P(x)$ be the polynomial of degree at most $m-1$ such that $f(a+r_i t) = P(a+r_i t)$, $(i=1, \cdots, m)$. Then if r is any rational number, $f(a+rt) = P(a+rt)$.*

Suppose that the lemma is not true. Then for some rational r_0, $f(a+r_0 t) \neq P(a+r_0 t)$. Let

$$Q(x) = c_0 + c_1 x + \cdots + c_m x^m, \quad (c_m \neq 0),$$

be the polynomial of degree m such that $Q(a+r_i t) = f(a+r_i t)$, $(i=0, \cdots, m)$. Set $\phi(x) = f(x) - Q(x)$; then $\phi(a+r_i t) = 0$, $(i=0, \cdots, m)$, and by §2 (c),

$$\Delta_h^m \phi(x) = \Delta_h^m f(x) - m! c_m \to - m! c_m$$

uniformly as $h \to 0$. Hence we can take a $\delta > 0$ so that $\Delta_h^m \phi(x) < 0$ or > 0 for all x in I and $h < \delta$ according as $c_m > 0$ or $c_m < 0$. As

* The case that $f(x)$ is continuous and $\Delta_h^m f(x) \equiv 0$ has been considered by Anghelutza, Mathematica (Cluj), vol. 6 (1932), pp. 1–7.

† Mathematische Annalen, vol. 60(1905), p. 461, equation (2).

the r_i are rational, there is a number r such that $rt<\delta$ and the numbers $a+krt$ (k an integer) include the numbers $a+r_it$. By Lemma 1, there are integers k' and k'' such that $\Delta_{rt}^m\phi(a+k'rt)\geqq 0$ and $\Delta_{rt}^m\phi(a+k''rt)\leqq 0$, a contradiction, proving the lemma.

5. *Proof of Theorem* 1. If we define $P(x)$ as in the lemma, a and t being rational, then $f(x)=P(x)$ at all rational points of I. If $f(x)$ is continuous, it follows that $f(x)\equiv P(x)$ in I. To complete the proof of the theorem, we must show that if $f(x)\not\equiv P(x)$, then $f(x)$ is not measurable.

Suppose there is a number a in I such that $f(a)\neq P(a)$. If $Q(x)$ is the polynomial of degree at most $m-1$ such that $f(x)=Q(x)$ at points $a, a+r_2, \cdots, a+r_m$, ($r_2, \cdots, r_m$ rational), then $f(x)=Q(x)$ at all points $a+r$, (r rational), by the lemma. Set $\sigma=|Q(a)-P(a)|$, and take $\delta>0$ so that

$$(7)\qquad |P(x)-P(a)|<\frac{\sigma}{4}, \qquad |Q(x)-Q(a)|<\frac{\sigma}{4}, \qquad (|x-a|\leqq\delta).$$

Take $\eta>0$ so that if $R(x)$ is any polynomial of degree at most $m-1$ such that $|R(i)|\leqq\eta$, $(i=1,\cdots,m)$, then $|R(0)|<\sigma/4$.* By a change of variable, we see then that for any x, h, and y_0, *if* $|R(x+ih)-y_0|\leqq\eta$, $(i=1,\cdots,m)$, *then*

$$|R(x)-y_0|<\sigma/4.$$

If $f(x)$ is measurable, there is a number y_0 such that the set of points E in $(a-\delta, a+\delta)$ (and in I) for which $|f(x)-y_0|\leqq\eta$ is of positive measure. Let I' be a closed subinterval of $(a-\delta, a+\delta)$ such that if $E_m=E\cdot I'$, then†

$$(8)\qquad \text{meas }(E_m)>\frac{m^2}{m^2+1}\text{ meas }(I').$$

Either $|y_0-P(a)|\geqq\sigma/2$ or $|y_0-Q(a)|\geqq\sigma/2$, say the latter. Let b be a number in I' such that $f(b)=Q(b)$, (take $b-a$ rational); then, using (7), $|f(b)-y_0|>\sigma/4$.

Let S be the set of numbers s such that $b+ms$ is in E_m, and E_i the set of numbers $b+is$, where s is in S, $(i=1,\cdots,m-1)$.

* That this can be done follows easily from the fact that $\Delta_1{}^m R(0)=0$; see §2 (c).

† See Hobson, *Functions of a Real Variable*, vol. 1, 3d ed., 1923, p. 194.

Set $E_i' = E_i \cdot E_m$, and let S_i be the set of numbers s in S such that $b + is$ is in E_i', $(i = 1, \cdots, m)$. There is no number s_0 in every S_i. For if there were, we would have simultaneously

$$(9) \quad |f(b + is_0) - y_0| \leqq \eta, \ (i = 1, \cdots, m), \ |f(b) - y_0| > \sigma/4.$$

Hence by our choice of η, if $R(x)$ is the polynomial of degree at most $m-1$ such that $R(b+is_0) = f(b+is_0)$, $(i = 1, \cdots, m)$, then $R(b) \neq f(b)$. But this contradicts Lemma 2.

Consequently, every number s of S is in some set $S - S_i$, and hence for some j, meas $(S - S_j) \geqq$ meas $(S)/m$. Therefore

$$\text{meas } (E_j - E_j') \geqq \frac{\text{meas } (E_j)}{m} \geqq \frac{\text{meas } (E_m)}{m^2}.$$

As E_m and $E_j - E_j'$ have no common points, this with (8) gives

$$(10) \qquad \text{meas } [E_m + (E_j - E_j')] \geqq \text{meas } (E_m)\left[1 + \frac{1}{m^2}\right] > \text{meas } (I').$$

But this contradicts the fact that $E_m + E_j$ lies in I'. Hence $f(x)$ is not measurable, and the theorem is proved.

THEOREM 2. *Let $f(x)$ be measurable in the closed interval $I = (\alpha, \beta)$. A necessary and sufficient condition* that $d^m f(x)/dx^m$ exist and equal $f_m(x)$ is that $\Delta_h^m f(x) \to f_m(x)$ uniformly in I.*

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* A direct proof of this theorem may be given as follows, as suggested by Birkhoff. The function $f(kh)$, ($h = 1/p$, k and p integral), may be expressed in terms of $\Delta_h^m f(x)$ by the formula

$$(12) \quad \begin{aligned} f(kh) = & \sum_{i=0}^{m-1} a_i (kh)^i \\ & + \frac{h^m}{(m-1)!} \sum_{l=0}^{k-1} (k-l-1)(k-l-2) \cdots (k-l-m+1) \Delta_h^m f(lh), \end{aligned}$$

where the a_i are determined so that $f(sph) = f(s)$, $(s = 0, \cdots, m-1)$. We solve a set of linear equations with a Vandermonde determinant, and find

$$(13) \quad \begin{aligned} a_i = & \sum_{s=0}^{m-1} \eta_{is} \left[f(s) \right. \\ & \left. - \frac{h^m}{(m-1)!} \sum_{l=0}^{sp-1} (sp-l-1) \cdots (sp-l-m+1) \Delta_h^m f(lh) \right], \end{aligned}$$

The necessity of the condition was proved in §2 (b). To prove the sufficiency, we show first that $f_m(x)$ is continuous. Take δ so small that $|\Delta_h^m f(x)-f_m(x)|<\epsilon/2$ for $|h|<\delta$. Take any two points x_1 and x_2 of I such that $x_2-x_1=mh$, $|h|<\delta$. Then $|\Delta_h^m f(x_1)-f_m(x_1)|<\epsilon/2$, $|\Delta_{-h}^m f(x_2)-f_m(x_2)|<\epsilon/2$. But $\Delta_{-h}^m f(x_2)=\Delta_h^m f(x_1)$; hence $|f_m(x_2)-f_m(x_1)|<\epsilon$.

Define the m-fold integral

$$(11)\qquad g(x) = \int_\alpha^x \cdots \int_\alpha^{x_3}\int_\alpha^{x_2} f_m(x_1)dx_1dx_2\cdots dx_m;$$

then $d^mg(x)/dx^m=f_m(x)$ in I, and by §2 (b), $\Delta_h^m g(x)\rightarrow f_m(x)$ uniformly in I. Set $\phi(x)=f(x)-g(x)$. Then $\Delta_h^m\phi(x)\rightarrow 0$ uniformly; hence $\phi(x)$ is a polynomial of degree at most $m-1$ (Theorem 1), and therefore $f(x)$ has a continuous mth derivative in I, and

$$\frac{d^m}{dx^m}f(x) = \frac{d^m}{dx^m}\phi(x)+\frac{d^m}{dx^m}g(x) = f_m(x).$$

THEOREM 3. *A necessary and sufficient condition that the function $f(x)=f_0(x)$ defined in I have a continuous mth derivative is that there exist functions $f_1(x),\cdots,f_m(x)$ such that* (2) *holds. In this case, $d^pf(x)/dx^p=f_p(x)$, $(p=1,\cdots,m)$.*

The necessity of the condition is a consequence of Taylor's formula. To prove the sufficiency, §2 (a) shows that $\Delta_h^p f(x)\rightarrow f_p(x)$ uniformly as $h\rightarrow 0$; consequently†, by Theorem 2, $d^pf(x)/dx^p$ exists and equals $f_p(x)$, $(p=1,\cdots,m)$.

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where the η_{is} are numerical coefficients. If in the above equations we replace p by $p_j=2^jp$, h by $h_j=h/2^j$, and k by $k_j=2^jk$, and set $r=k_jh_j=kh$, then as $\Delta_h^m f(x)\rightarrow f_m(x)$ uniformly, passing to the limit $j=\infty$ gives

$$(14)\qquad \begin{aligned}(r) = \sum_{i=0}^{m-1} r^i\sum_{s=0}^{m-1}\eta_{is}\left[f(s)-\frac{1}{(m-1)!}\int_0^s(s-t)^{m-1}f_m(t)dt\right]\\ +\frac{1}{(m-1)!}\int_0^r(r-t)^{m-1}f_m(t)dt,\end{aligned}$$

for any rational $r>0$. If $f(x)$ is continuous, this is true for all $r>0$ and, the theorem follows immediately on differentiating. (The restriction $r>0$ may of course be replaced by the restriction $r>-c$ for any c.)

† A direct proof of this theorem is given by the author in a paper entitled *Differentiable functions defined in closed sets I*, Theorem IV.