NOTE ON THE ORTHOGONALITY OF TCHEBYCHEFF POLYNOMIALS ON CONFOCAL ELLIPSES*

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In the study of polynomial expansions of analytic functions in the complex plane, two different definitions of orthogonality are current:

$$(1)\qquad \int_C n(z)p_j(z)p_k(z)\,dz = 0, \text{ or } \int_C p_j(z)q_k(z)dz = 0, \quad (j \neq k),$$

$$(2)\qquad \int_C n(z)p_j(z)\overline{p_k(z)}\,|\,dz\,| = 0, \qquad (j \neq k).$$

Definition (1) in one form or the other (the second form of (1) may be called biorthogonality) is of frequent use, for instance in connection with the Legendre polynomials,† and has the great advantage that if the functions involved are analytic, the contour of integration C may be deformed without altering the orthogonality property. Definition (2) is of importance—indeed inevitable—when one wishes to study approximation on C in the sense of least squares, and it is entirely with definition (2) that we shall be concerned in the present note. More explicitly, condition (2) may be described as orthogonality with respect to the norm function $n(z)$, ordinarily chosen as continuous and positive or at least non-negative on C.

An illustration of (1), where C is the unit circle $|z| = 1$, is the set of functions $1, z, z^2, \cdots, n(z) \equiv 1$. An illustration of (2), where C is the unit circle, is the set of functions $\cdots, z^{-2}, z^{-1}, 1, z, z^2, \cdots, n(z) \equiv 1$:

$$\int_C z^j \bar{z}^k\,|\,dz\,| = \int_C \frac{z^j}{z^k}\frac{dz}{iz} = 0, \qquad (j \neq k).$$

The connection of orthogonality in the sense of (2) with ap-

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† The reader may refer to Heine, *Kugelfunktionen*, 1878; Darboux, Journal de Mathématiques, (3), vol. 4 (1878), pp. 5–56, and pp. 377–416; Geronimus, Transactions of this Society, vol. 33 (1931), pp. 322–328.

proximation in the sense of least squares has long been known.* The application to polynomial expansions on curves in the complex plane is due to Szegö† in the case $n(z)\equiv 1$ and to Walsh‡ in the more general case. The most important theorem which concerns us here is the following, which is due to Szegö ($n(z)\equiv 1$, C analytic), Smirnoff ($n(z)\equiv 1$, C rectifiable and subject to auxiliary condition), and Walsh, loc. cit., ($n(z)$ positive and continuous, C rectifiable).

Let C be a rectifiable Jordan curve and let the function $w=\phi(z)$ map the exterior of C onto the exterior of $|w|=1$ so that the points at infinity correspond to each other. Let the curve $|\phi(z)|=R>1$ be generically denoted by C_R.

If the function $f(z)$ is analytic interior to C_ρ but has a singularity on C_ρ, and if

$$
f(z)\sim \sum a_k p_k(z),
$$
$$
(3)\qquad a_k\int_C n(z)p_k(z)\overline{p_k(z)}\,|dz| = \int_C n(z)f(z)\overline{p_k(z)}\,|dz|,
$$

is the formal expansion of $f(z)$ in terms of the polynomials $p_k(z)$ of respective degrees k orthogonal on C with respect to the function $n(z)$ positive and continuous on C, then series (3) *converges to $f(z)$ interior to C_ρ, uniformly on any closed point set interior to C_ρ, and converges uniformly in no region containing in its interior a point of C_ρ.*

There is no series other than (3) of the form $\sum b_k p_k(z)$ which converges to $f(z)$ uniformly on C.

This theorem is valid also in the limiting case that C is a line segment or other rectifiable Jordan arc. The case that C is a line segment has long been studied and by numerous writers; for instance the weight function $n(z)\equiv 1$ leads to expansions in Legendre polynomials treated for complex values of the argument by C. Neumann in 1862. When C is a line segment, the curves C_R are ellipses whose foci are the ends of the segment.

The theorem naturally raises the question as to whether (3) can be both the formal expansion of $f(z)$ on C and the formal expansion of $f(z)$ on some $C_{\rho'}$, $\rho'<\rho$, or in other words whether

* See for instance Kowalewski, *Determinantentheorie*, 1909, p. 335.

† Mathematische Zeitschrift, vol. 9 (1921), pp. 218–270.

‡ Transactions of this Society, vol. 32 (1930), pp. 794–816.

the same set of polynomials $p_k(z)$ can result from orthogonalization (in the sense corresponding to (2)) of the set $1, z, z^2, \cdots$, on two different curves. An obvious illustration is the case that C is the unit circle. The polynomials $1, z, z^2, \cdots$ are orthogonal with weight function unity on *every* circle whose center is the origin, and the formal expansion (3) of a function $f(z)$ analytic at the origin is the same on every such circle containing on or within it no singularity of $f(z)$.

It is the object of the present note to show that *the Tchebycheff polynomials found by orthogonalization of* $1, z, z^2, \cdots$ *on the line segment* $C: -1 \leqq z \leqq +1$ *with respect to the norm function* $(1-z^2)^{-1/2}$ *are also orthogonal with respect to suitable norm functions on all the corresponding curves* C_R, *which are ellipses with the common foci* $(-1, +1)$.

We shall find it more convenient to transform our problem to the w-plane. The exterior of $C: -1 \leqq z \leqq +1$ is transformed onto the exterior of $\gamma: |w| = 1$ by the transformation

$$z = \frac{1}{2}\left(w + \frac{1}{w}\right), \tag{4}$$

so that the points at infinity correspond to each other. Let the polynomials $p_0(z) \equiv 1, p_1(z), p_2(z), \cdots$ result from orthogonalizing on γ the linearly independent set $1, z, z^2, \cdots$ with norm function unity. We shall prove that the polynomials $p_k(z)$ form an orthogonal set on $\Gamma: |w| = R > 1$ with norm function unity. For the sake of reference we write the equations

$$\begin{aligned} z &= \frac{1}{2}\left(w + \frac{1}{w}\right), \quad z^2 = \frac{1}{4}\left(w^2 + 2 + \frac{1}{w^2}\right), \\ z^3 &= \frac{1}{8}\left(w^3 + 3w + \frac{3}{w} + \frac{1}{w^3}\right), \cdots, \end{aligned} \tag{5}$$

whence, on Γ,

$$\begin{aligned} \bar{z} &= \frac{1}{2}\left(\frac{R^2}{w} + \frac{w}{R^2}\right), \quad \bar{z}^2 = \frac{1}{4}\left(\frac{R^4}{w^2} + 2 + \frac{w^2}{R^4}\right), \\ \bar{z}^3 &= \frac{1}{8}\left(\frac{R^6}{w^3} + \frac{3R^2}{w} + \frac{3w}{R^2} + \frac{w^3}{R^6}\right), \cdots. \end{aligned} \tag{6}$$

We need merely prove that the polynomial $p_n(z)$ is orthogonal to each of the polynomials $p_0(z), p_1(z), \cdots, p_{n-1}(z)$, and it is sufficient to show that $p_n(z)$ is orthogonal to each of the functions $1, z, \cdots, z^{n-1}$. We shall prove this by induction.

We have by hypothesis

$$\int_\gamma p_n(z)\,|\,dw\,| = 0, \tag{7}$$

and we show first that we have also

$$\int_\Gamma p_n(z)\,|\,dw\,| = 0. \tag{8}$$

Direct computation gives us $|\,dw\,| = R\,dw/(iw)$ for w on Γ. When this substitution is made in (7) and (8), it is seen that the integral in (8) is except for the factor R precisely the integral in (7); the function $p_n(z)$ has no singularities in the w-plane except at the origin and at infinity. Hence equation (8) follows at once.

Let us now suppose

$$\int_\Gamma p_n(z)\,|\,dw\,| = 0, \quad \int_\Gamma p_n(z)\bar{z}\,|\,dw\,| = 0, \cdots, \tag{9}$$

$$\int_\Gamma p_n(z)\bar{z}^{k-1}\,|\,dw\,| = 0, \qquad (k-1 < n-1).$$

We are to prove that

$$\int_\Gamma p_n(z)\bar{z}^k\,|\,dw\,| = 0. \tag{10}$$

From inspection of equations (6) and by virtue of equations (9), it follows that the integral in (10) can be written

$$\frac{1}{2^k}\int_\Gamma p_n(z)\left(\frac{R^{2k}}{w^k} + \frac{w^k}{R^{2k}}\right)\frac{Rdw}{iw}\,. \tag{11}$$

If we consider the various terms in $p_n(z)$ as expressed by means of (5), and omit such of those terms as obviously make no contribution to the integral (11), we see that (11) can be written

$$A_{nk}\int_\Gamma \left(w^k + \frac{1}{w^k}\right)\left(\frac{R^{2k}}{w^k} + \frac{w^k}{R^{2k}}\right)\frac{Rdw}{iw} \tag{12}$$

$$= 2\pi A_{nk}R\left[R^{2k} + \frac{1}{R^{2k}}\right],$$

where A_{nk} is a suitable constant independent of R, easy to compute in terms of the coefficients of the powers of z in $p_n(z)$. By hypothesis the polynomials $p_n(z)$ are orthogonal on the circle γ. Hence the integrals corresponding to (10), (11), and (12) vanish for $R=1$. Thus we have $A_{nk}=0$, so (10) is established and the orthogonality on Γ of the set $p_k(z)$ with respect to the norm function unity is completely proved.

It remains to study the norm function in the z-plane, and to identify our present polynomials with the polynomials of Tchebycheff. From (4) we have for $|w|=R$,

$$dz = \frac{1}{2}\left(1 - \frac{1}{w^2}\right) dw, \qquad \left|\frac{dw}{dz}\right| = \frac{2R}{\left|w - \dfrac{1}{w}\right|} = \frac{R}{|(1 - z^2)^{1/2}|}.$$

The circle $|w|=1$ corresponds to the segment $-1 \leqq z \leqq +1$ (counted twice, or in the study of orthogonality, only once if we prefer, for the norm function is single-valued on the segment). We clearly have

$$\int_{-1}^{1} n(z)p_n(z)\overline{p_k(z)}\,|dz| = \int_{\gamma} n(z)p_n(z)\overline{p_k(z)}\,|dw|\left|\frac{dz}{dw}\right|,$$

so the corresponding norm function on the segment $-1 \leqq z \leqq +1$ is $1/|(1-z^2)^{1/2}|$. The norm function on an arbitrary ellipse whose foci are $+1$ and -1 is $R/|(1-z^2)^{1/2}|$, where the ellipse is represented by $|z-1|+|z+1|=R+1/R$, $R>1$. The norm function on any curve can be modified by any non-vanishing constant factor, so if we prefer we can still express the norm function on any ellipse C_R as $|1-z^2|^{-1/2}$.

In the present note we have studied orthogonality on a particular set of confocal ellipses; a simple transformation yields the analogous results for orthogonality of corresponding polynomials on any set of confocal ellipses.

The writer is not aware of any case other than the present one, where orthogonalization in the sense of (2) of the set $1, z, z^2, \cdots$, on a curve C_1 with respect to a norm function $n_1(z)$, is equivalent to orthogonalization of that set on another curve C_2 with respect to a norm function $n_2(z)$, except where C_1 and C_2 are concentric circles.

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