ear dimensions. We say that, for two spaces of type (F), (1) $\dim_l E \leqq \dim_l E_1$ if E is isomorphic with a linear closed sub-space of E_1; that $\dim_l E = \dim_l E_1$ if also (2) $\dim_l E_1 \leqq \dim_l E$, while $\dim_l E < \dim_l E_1$ if (1) is satisfied but (2) is not. If neither (1) nor (2) is satisfied, the linear dimensions of E and E_1 are said to be incomparable. It should be observed that, while two isomorphic spaces are of the same linear dimensions, there exist separable spaces (B) which are not isomorphic although their linear dimensions are the same. Special attention is given here to spaces (L^p) and (l^p). It is shown that if $\dim_l(L^p) = \dim_l(l^q)$, where $p, q > 1$, then $p = q$; if $1 < p < 2 < q$, then the linear dimensions of (L^p) and (l^q) are incomparable; if $1 < p \neq 2$, then $\dim_l(L^2) < \dim_l(L^p)$; if $\dim_l(L^p) < \dim_l(L^q)$, where $p, q > 1$, then $p = q = 2$; the condition $p = q = 2$ is necessary and sufficient that $\dim_l(L^p) = \dim_l(l^q)$; if $1 < p \neq 2$, then $\dim_l(L^p) > \dim_l(l^p)$. The Appendix (18 pp.) contains an additional discussion of the weak convergence of elements and of functionals. The book closes with 21 pages of additional remarks containing historical information and statements of various generalizations of the results in the text, as well as numerous unsolved problems. These remarks reveal that the whole subject is in a state of vigorous development. The usefulness of many of these remarks would be considerably greater if they contained hints indicating how the results in question have been obtained. The exposition as a rule is clear and detailed, although not easy in several places. The number of misprints (of which only two are corrected in the list of Errata) is not negligible.

In conclusion the reviewer may express his conviction that Banach's monograph should occupy a permanent and honorable place on the desk of every one who is interested in the theory of linear operations, to be replaced only by subsequent, corrected and augmented editions, which undoubtedly will follow before long.

J. D. TAMARKIN

SAKS ON INTEGRATION

Théorie de l'Intégrale. By Stanislaw Saks. Warsaw, Monografje Matematyczne, Vol. II, 1933. vii+290 pp.

The present volume is the second one of the series Monografje Matematyczne. It is a translation, entirely revised and augmented by several important chapters, of the author's Polish book, *Zarys Teorji Calki*, Warsaw, 1930. It fills in a serious gap in the literature of the real function theory, and of the theory of differentiation and integration, which has been acutely felt during all the recent period of vigorous growth and development of these disciplines. While Hermite, together with a large part of his contemporaries and immediate successors, including Poincaré, contemplated with horror the pathological cases of functions without derivatives, the ideas and methods created precisely for handling such bad cases turned out to be extremely fruitful and indispensable for the treatment of most classical and venerated problems of analysis. Far from bringing in the feared anarchy and disorder, they have allowed us in many instances to reach a harmony and completeness of results which were entirely out of the reach of older classical methods. The problems which require essentially a fundamental knowledge of the most modern chapters of the real function theory are "everywhere dense" in the field of analysis, and the mathematician who wants to make any headway in the theory of functions of a complex variable, of differential and integral equations, of Fourier series, and the like, will find himself badly handicapped if he is not well equipped on the side of the real function theory, and particularly of the theory of differentiation and integration. But these are precisely the chapters which so far have been lacking an adequate treatment in the literature. Except for the treatise of Hobson, which still holds its honorable place, we are unable to name any other book or monograph where a serious attempt has been made to unify the numberless results scattered throughout all mathematical periodicals and to give a general picture of the present state of the subject. Such an attempt, and a fully successful one, is represented by the book of Saks. Its main characteristic unifying feature is the systematic use, from the very beginning, of the notion of additive functions of sets. This allows him to bring in, with greatest clearness and economy of thought and notation, the *descriptive* theory of various integrals he discusses. At the same time it leads very naturally to fundamental problems of the modern differentiation theory, and to an extensive use of Dini's derived numbers and of approximate derivatives and derived numbers, as well as to the question of the unique determination of an additive function of sets by means of its derived numbers of various sorts. The systematic development of these notions constitutes the very nerve of the book. A mathematician who will not avail himself of the rich source of information brilliantly presented in Saks' book will deprive himself of the use of valuable ideas and tools which would be of help to him during all his "analytical" career. In publishing the present monograph and the preceding one by Banach, the Monografje Matematyczne have set up a high standard which will be difficult to match. We congratulate them on this happy start and we wish to attract to this important undertaking all attention of the mathematical world, which it justly deserves. The next books which are announced in this series are: K. Kuratowski, *Topologie* I (Vol. 3); W. Sierpinski, *Hypothèse du Continu* (Vol. 4), and A. Zygmund, *Trigonometric series* (Vol. 5).

The first five chapters of Saks's book treat of the theory of functions of bounded variation, Lebesgue measure and Lebesgue integral, together with a short discussion of the Riemann-Darboux-Stieltjes integral. Without losing anything in simplicity or elegance but gaining in generality and clearness, the author discusses functions of several variables, rather than those of a single variable. An excellent example of application of the foregoing theory presents itself in the theory of areas of surfaces given in explicit form $z=f(x, y)$, as developed by Lebesgue, Geöcze, Tonelli, and Radó. The exposition of this beautiful theory has never been given before in any book; it occupies Chapter 6. Chapter 7 deals with the notion of a Perron integral and its relationship with the integral of Lebesgue. A generalization of the majorant and minorant functions introduced by La Vallée Poussin plays an important role here. Chapters 8 (*Functions of generalized bounded variation*) and 9 (*Theorems on derived numbers*) contain material which is important in itself and at the same time prepares the ground for the theory of the Denjoy integral. Chapter 9 contains a very careful and complete discussion of the problem of unique determination of functions of various classes by means of their derivatives or derived numbers of various sorts. Chapter 10 is devoted to the integral of Denjoy. The author gives here a remarkably clear and concise exposition of this delicate theory which he developed in his memoir published in the Fundamenta Mathematicae. Chapter 11 contains a careful exposition of some most recent results of the theory of the approximate and total differentials of a function of two variables. Some of the results which are discussed here are entirely new. The end of this chapter is of a particular significance. It contains a beautiful proof by Menchoff (never published before) of the theorem to the effect that if $f(z)$ is continuous in a domain and admits of partial derivatives finite everywhere (or even only at all points except for a denumerable set), and if these derivatives satisfy the Cauchy-Riemann equations, then $f(z)$ is analytic. This discussion represents a striking example of what can be achieved in most classical problems of analysis by methods of the modern real function theory. The last Chapter is devoted to a rapid but fairly complete exposition of the theory of the Lebesgue integral in abstract spaces. This theory should have found its way long ago into the treatises on the real function theory. The introduction of such a chapter in Saks's book is a welcome innovation. At the end of the book there is an interesting note by Banach containing a simplified exposition of the theory of measure of Haar, which proved to be of extreme importance for the theory of continuous groups. The book is supplied with a long (12 pp.) bibliography, interesting in itself. The book is very well printed and is practically free from misprints (except for those corrected in the Errata). In conclusion we may state our conviction that Saks's book is eminently suited to be used in a graduate course on the real function theory. This opinion has been checked by the reviewer's personal experience.

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