BANACH ON LINEAR OPERATIONS

Théorie des Opérations Linéaires. By Stefan Banach. Warsaw, Monografje Matematyczne, Vol. I, 1932. vii+254 pp.

The present book, the first of the series published by the Monografje Matematyzcne, is an enlarged and considerably modified translation of the *Teorja Operacyi* (in Polish) by Banach. It represents a noteworthy climax of long series of researches started by Volterra, Fredholm, Hilbert, Hadamard, Fréchet, F. Riesz, and successfully continued by Steinhaus, Banach, and their pupils. In a short review it is impossible to give even an approximate idea of the richness and importance of the material, entirely new to a large extent, which is gathered in this book. Roughly speaking the book treats of the field which was touched upon by Hildebrandt in his excellent address on linear functional transformations in general spaces,* but naturally is considerably more extensive and intensive. The theory of linear operations is a fascinating field in itself but its importance is still more emphasized by numerous beautiful applications to various problems of analysis and function theory (real and complex). A number of these applications are contained in the book; a still larger number of them, increasing from year to year, are being inspired by the ideas developed in the book. An attentive reader will find a constant source of inspiration in this monograph which undoubtedly will exercise a great influence on the further development of the subject. A description of the contents follows.

In the Introduction (19 pp.) the author recalls some basic properties of Lebesgue and Stieltjes integrals, and of metric spaces. Particular attention is given to subsets of such spaces, which either are Borel measurable or possess the property of Baire, and also to Borel measurable operations transforming one metric space into another. In Chapter 1 (*Groupes*, 6 pp.) the author specializes the metric space under consideration by adding the usual postulates for groups (with respect to the operation "+", with inverse designated by "−") and also the two postulates concerning limits:

$$x_n \to x \quad \text{implies} \quad (-x_n) \to (-x);$$

$$x_n \to x \text{ and } y_n \to y \quad \text{implies} \quad (x_n + y_n) \to (x + y).$$

It is proved that every subgroup which possesses Baire's property and is of the second category coincides necessarily with the whole space (assumed to be complete). The additive and linear (additive+continuous) operations are introduced at this early stage. It is proved that the set where a sequence $\{U_n(x)\}$ of linear operations converges, or even where it is only bounded, is either the whole space or a set of the first category (the space is assumed to be connected). Chapter 2 (*Espaces vectoriels généraux*, 9 pp.) deals with linear spaces and additive operations defined on such spaces. The central point here is an important theorem concerning a possibility of extending to the whole space

* This Bulletin, vol. 37 (1931), pp. 185–212.

an additive functional (a real-valued operation) defined on a linear sub-space. Theorems of this kind originated with F. Riesz (in the case of space of continuous functions) and the theorem in question was proved independently by Hahn. As an application it is shown that every bounded sequence has a "limit" which is distributive and possesses some other properties of the usual limit, and is always enclosed between the limits of indetermination of the sequence. An analogous application is made to show that an arbitrary bounded function may be supplied with an "integral," as well as every bounded linear set with a "measure." In Chapter 3 (*Espaces du type* (F), 18 pp.) a metric complete space is specialized to be linear and to satisfy three additional postulates:

$$(x, y) = (x - y, 0);$$

$$h_n \to 0 \quad \text{implies} \quad h_n x \to 0, \quad \text{where } \{h_n\} \text{ is a sequence of constants};$$

$$x_n \to 0 \quad \text{implies} \quad h x_n \to 0, \quad \text{where } h \text{ is a constant}.$$

A fundamental property of linear operations (transforming such a space E of type (F) into a set of another space E_1, also of type (F)) is that the range of a linear operation is either a set of the first category in E_1 or E_1 itself, while, in the case when the transformation (linear) is one to one, it is also bicontinuous. This chapter contains important applications to the theory of continuous non-differentiable functions, the continuity of solutions of partial differential equations, and the solution of certain systems of infinitely many equations of a general type.

The next specialization is furnished by *normed* spaces which are metric and linear with the metric $\|x\|$ satisfying the postulates

$$\|x + y\| \leqq \|x\| + \|y\|, \quad \|cx\| = |c| \, \|x\|,$$

where c is a constant. A normed space, if it is complete, is designated as a space (B). Such spaces have been also introduced by Wiener, independently and almost simultaneously with Banach. Chapter 4 (*Espaces normés*, 25 pp.) deals with normed spaces in general. Here we find a determination of the form of the most general linear functional in various important special spaces, such as spaces of all convergent sequences, (c); of sequences $x = \{\xi_n\}$ with

$$\sum |\xi_n|^p < \infty, \quad (l^p);$$

of continuous functions, (C); of measurable functions $x(t)$ with

$$\int_0^1 |x(t)|^p dt < \infty, \quad (L^p).$$

We find also interesting applications to the theory of approximation (in various spaces) to a given function by means of linear combinations of functions of a given set, and to the moment problem. Chapter 5 (*Espaces du type* (B), 18 pp.) deals especially with normed complete spaces. Here we find the fundamental result that if $\{U_n(x)\}$ is a sequence of linear operations, bounded at each point of the second category of a space E of type (B), then the sequence of norms $\{\|U_n\|\}$ is bounded. This theorem is the most general among similar results of more or less special nature obtained by various other authors. Some interesting applications to the comparative study of definitions of summability are also found in this chapter. We mention also some results of importance for the theory of Fredholm integral equations with kernels of "bounded" type, which are out of the reach of the more powerful theory developed by von Neumann and Stone. Chapter 6 (*Opérations totalement continues et associées*, 10 pp.) introduces the notions of the adjoint (or conjugate) operation, and of a completely continuous operation, whose importance is so well known in the more special theories of integral equations and bilinear forms. Chapter 7 (*Suites biorthogonales*, 9 pp.) contains various applications to the general theory of biorthogonal expansions. Of a particular elegance are the results concerning factor-sequences leaving invariant the "class" (L^p) or (C) of a Fourier series. Chapter 8 (*Fonctionnelles linéaires dans les espaces du type* (B), 18 pp.) is devoted to a more detailed study of various sets of linear functionals in spaces (B). Three important notions are introduced and discussed, those of sets of functionals which are regularly closed, transfinitely closed, and weakly closed. The two first are shown to be equivalent for every space (B), while all three are equivalent in case the space (B) is also separable. Conditions are given in order that a given set of functionals coincide with the space of all the functionals (the conjugate space). Chapter 9 (*Suites faiblement convergentes d'éléments*, 12 pp.) gives a discussion of weak convergence, an important notion introduced in special cases by F. Riesz. After all these preparations the author can easily develop his beautiful theory of linear functional equations in Chapter 10 (*Équations fonctionnelles linéaires*, 20 pp.). As very special instances of applications of results of this chapter we may mention the theory of non-singular, and some singular, Fredholm integral equations and of Hilbert's theory of limited and completely continuous bilinear forms. The power of the method (the basic ideas of which were previously introduced by F. Riesz and by Hildebrandt) is due to the fact that it is free from any special way of representation of operations in question, such as integrals or matrices, but deals directly with operations themselves.

The last two chapters treat of entirely different questions. These chapters, as well as the Appendix at the end of the book, were not present in the Polish edition of the book. Chapter 11 (*Isométrie, equivalence, isomorphie*, 28 pp.) discusses various types of correspondences (all of them one to one) between two spaces E and E_1 of type (B). These spaces are called isometric if the correspondence preserves the norm, isomorphic if this correspondence is linear, and equivalent if they are isomorphic and isometric. Some of these notions can be of course extended to more general spaces. A rotation of E about a point x_0 is defined as an isometric transformation of E into itself, which leaves invariant the point x_0. Various beautiful results are found in this chapter, of which we mention only three: Every isometric transformation of a linear normed space into another space of the same type is linear, provided it transforms the zero element into the zero element (Mazur and Ulam). Let Q and Q_1 be two metric complete and compact sets. A necessary and sufficient condition that they be homeomorphic is that the spaces (C_Q) and (C_{Q_1}) of real-valued continuous functions defined over Q and Q_1 respectively be isometric. Every separable space of type (B) is equivalent to a closed linear sub-space of the space (C) of continuous functions; every metric separable space is isometric with a sub-space of (C). The expressions for the most general rotations in spaces (C), (c), (L^p), (l^p) are also found in this chapter. Chapter 12 (*Dimensions linéaire*, 15 pp.) introduces a method of comparing various spaces according to their *lin-*

ear dimensions. We say that, for two spaces of type (F), (1) $\dim_l E \leqq \dim_l E_1$ if E is isomorphic with a linear closed sub-space of E_1; that $\dim_l E = \dim_l E_1$ if also (2) $\dim_l E_1 \leqq \dim_l E$, while $\dim_l E < \dim_l E_1$ if (1) is satisfied but (2) is not. If neither (1) nor (2) is satisfied, the linear dimensions of E and E_1 are said to be incomparable. It should be observed that, while two isomorphic spaces are of the same linear dimensions, there exist separable spaces (B) which are not isomorphic although their linear dimensions are the same. Special attention is given here to spaces (L^p) and (l^p). It is shown that if $\dim_l(L^p) = \dim_l(l^q)$, where $p, q > 1$, then $p = q$; if $1 < p < 2 < q$, then the linear dimensions of (L^p) and (l^q) are incomparable; if $1 < p \neq 2$, then $\dim_l(L^2) < \dim_l(L^p)$; if $\dim_l(L^p) < \dim_l(L^q)$, where $p, q > 1$, then $p = q = 2$; the condition $p = q = 2$ is necessary and sufficient that $\dim_l(L^p) = \dim_l(l^q)$; if $1 < p \neq 2$, then $\dim_l(L^p) > \dim_l(l^p)$. The Appendix (18 pp.) contains an additional discussion of the weak convergence of elements and of functionals. The book closes with 21 pages of additional remarks containing historical information and statements of various generalizations of the results in the text, as well as numerous unsolved problems. These remarks reveal that the whole subject is in a state of vigorous development. The usefulness of many of these remarks would be considerably greater if they contained hints indicating how the results in question have been obtained. The exposition as a rule is clear and detailed, although not easy in several places. The number of misprints (of which only two are corrected in the list of Errata) is not negligible.

In conclusion the reviewer may express his conviction that Banach's monograph should occupy a permanent and honorable place on the desk of every one who is interested in the theory of linear operations, to be replaced only by subsequent, corrected and augmented editions, which undoubtedly will follow before long.

J. D. TAMARKIN

SAKS ON INTEGRATION

Théorie de l'Intégrale. By Stanislaw Saks. Warsaw, Monografje Matematyczne, Vol. II, 1933. vii+290 pp.

The present volume is the second one of the series Monografje Matematyczne. It is a translation, entirely revised and augmented by several important chapters, of the author's Polish book, *Zarys Teorji Calki*, Warsaw, 1930. It fills in a serious gap in the literature of the real function theory, and of the theory of differentiation and integration, which has been acutely felt during all the recent period of vigorous growth and development of these disciplines. While Hermite, together with a large part of his contemporaries and immediate successors, including Poincaré, contemplated with horror the pathological cases of functions without derivatives, the ideas and methods created precisely for handling such bad cases turned out to be extremely fruitful and indispensable for the treatment of most classical and venerated problems of analysis. Far from bringing in the feared anarchy and disorder, they have allowed us in many instances to reach a harmony and completeness of results which were entirely out of the reach of older classical methods. The problems