NOTE ON THE LOCATION OF THE CRITICAL POINTS OF GREEN'S FUNCTION*

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1. *Introduction.* The Green's function for a multiply connected plane region enters into many investigations, so that the nature of the equipotential curves, and in particular of their singularities, is of some importance. The writer's immediate interest in the subject, for instance, arises from the study of approximation of analytic functions by polynomials; the degree of best approximation and regions of convergence of sequences of polynomials of best approximation can be described directly and simply in terms of the equipotential curves of a certain Green's function.† For an arbitrary plane region the singular points of the equipotential curves are precisely the critical points of the Green's function, that is, the points where both first partial derivatives of Green's function vanish.

The results on the location of the critical points of the Green's function which we here prove are analogous to and proved by the use of certain well known results on the location of the roots of the derivative of a polynomial, and are not difficult to establish. However, they seem not to be known even among those versed in potential theory, and are therefore set forth here.

2. *The Green's Function.* Let R be a region of the plane, and let $P:(a, b)$ be a point of R. The Green's function $G(x, y)$ for R with pole at P is the function which is harmonic‡ interior to R except at P, continuous in the corresponding closed region except at P, zero on the boundary of R, and in some neighborhood of P can be expressed as $U(x, y) - \frac{1}{2}\log[(x-a)^2+(y-b)^2]$, where $U(x, y)$ is harmonic at P. The Green's function need not exist for an arbitrary region, but, if existent, is unique. The Green's function for a region is invariant (except for a constant factor) under conformal transformation. In particular this invariant property

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† Walsh and Russell, forthcoming in the Transactions of this Society.

‡ A function is *harmonic at a point* if in some neighborhood of that point it is continuous together with its first and second partial derivatives and satisfies Laplace's equation. A function is *harmonic in a region* if it is harmonic at every point of that region. A *region* is an open connected point set.

gives a meaning for $G(x, y)$ when P is at infinity. The Green's function $G(x, y)$ for an infinite region R with pole P at infinity is harmonic interior to R except at infinity, continuous in the corresponding closed region except at P, zero on the boundary of R, and in the neighborhood of P can be expressed in the form $U(x, y)+\frac{1}{2}\log(x^2+y^2)$, where $U(x, y)$ is harmonic at P.

The Green's function admits the following physical interpretation. Let the boundary B of R (considered of electrically conducting material) be kept at potential zero. The potential at an arbitrary point of R due to a unit charge at P and the charge on B induced by it is $G(x, y)$. The points of equilibrium in the corresponding field of force are the critical points of $G(x, y)$. The induced charge on B is non-positive at all points of B (if this physical interpretation is justified and B is smooth), for $G(x, y)$ is everywhere positive interior to R, and approaches zero as (x, y) approaches B.

3. *Approximation by Lemniscates.* Rather than give proofs of our results based on the physical interpretation just given, which is not difficult (compare the references given below), we prefer to give more rigorous proofs based on analytic methods.

We mean by a *lemniscate* a locus of the form

$$(1)\qquad |p(z)| = \mu > 0, \qquad p(z) = (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_\sigma).$$

A lemniscate thus consists of one or more mutually exterior contours, although a finite number of exceptional points may belong to several of these contours. The points α_k lie interior to the contours. By the *interior of the lemniscate* we mean the interiors of the contours, that is, the points at which we have $|p(z)|<\mu$. *If mutually exterior arbitrary Jordan curves* $C_1, C_2, \cdots, C_\nu$ *are given, and arbitrarily small annular regions containing these curves, then there exists a lemniscate of form* (1) *which contains the* C_k *in its interior and consists of precisely* ν *Jordan curves lying one in each of those regions.**

If R is an arbitrary infinite region whose boundary B is finite, there exists a sequence of lemniscates L_n each interior to its

* Proved in the case $\nu=1$ by Hilbert, Göttinger Nachrichten, 1897, pp. 63–70, and in the general case by Walsh and Russell, loc. cit. Compare also Faber, Münchner Berichte, 1922, pp. 157–178, and Pólya and Szegö, Journal für Mathematik, vol. 165 (1931), pp. 4–49, with the references there given.

predecessor, each lying in R, and such that each point of R is interior to only a finite number of the L_n. It is then well known that the Green's function $G_n(x, y)$ with pole at infinity for the exterior of L_n approaches uniformly (in any closed region interior to R) the Green's function $G(x, y)$ with pole at infinity for R. It is also true, and follows from the method of proof of approximation by lemniscates, that if R is symmetric with respect to a point or line, then the lemniscates L_n can be chosen symmetric with respect to that point or line.

4. *Critical Points of the Green's Function.* For the exterior: $|p(z)| > \mu$ of the lemniscate (1), the Green's function with pole at infinity is $(1/\sigma) \log |p(z)/\mu|$, where σ is the degree of $p(z)$, as may be verified immediately. The critical points of this function are precisely the roots of $p'(z)$, concerning the location of which many theorems are known, in terms of the location of the α_k.

Let us now show that *for the region R of* §3, *each critical point of $G(x, y)$ in R is the limit of critical points of the $G_n(x, y)$.* Let P_1 be a critical point of $G(x, y)$ interior to R. There exists some circle C whose center is P_1 containing on or within it no point of B. All the points of L_n, for n sufficiently large, lie exterior to C, so for n sufficiently large all the functions $G_n(x, y)$ are harmonic interior to C. Let $H_n(x, y)$ be the function conjugate to $G_n(x, y)$ which vanishes at P_1; then $H_n(x, y)$, if suitably defined, is harmonic and single-valued on and within C. Let us set $F_n(z) = G_n(x, y) + iH_n(x, y)$; the sequences $\partial G_n/\partial x$, $\partial G_n/\partial y$, $H_n(x, y)$, $F_n(z)$, $F_n'(z)$ converge uniformly in an arbitrary circle C' interior to C with center at P_1, and these sequences have as their respective limits interior to C the functions $\partial G/\partial x$, $\partial G/\partial y$, $H(x, y)$, $F(z)$, $F'(z)$, where $H(x, y)$ is the function conjugate to $G(x, y)$ which vanishes at P_1 and where we have $F(z) = G(x, y) + iH(x, y)$. The function $F'(z)$ vanishes at P_1, by hypothesis. Then by a well known theorem due to Hurwitz, since $F'(z)$ does not vanish identically, the function $F_n'(z)$ vanishes in an arbitrary neighborhood of P_1, for n sufficiently large. Our proof is complete.

5. *Analog of the Gauss-Lucas Theorem.* The Gauss-Lucas theorem is: *If $p(z)$ is an arbitrary polynomial, then all roots of the derivative $p'(z)$ lie in the smallest convex region which contains the roots of $p(z)$.*

The analog which we shall prove is as follows.

THEOREM 1. *If R is an infinite region whose boundary is finite, then all the critical points of the Green's function (if existent) for R with pole at infinity lie in the smallest convex region which contains the boundary B of R.*

If all the points of B lie *interior* to a convex region K, all the lemniscates L_n (for n sufficiently large) also lie interior to K. All the roots of the polynomials defining those lemniscates lie interior to the respective L_n and hence interior to K, so (by the Gauss-Lucas theorem) all the critical points of the $G_n(x, y)$ lie interior to K and all their limit points lie in the corresponding closed region. Theorem 1 follows at once.

Another statement of Theorem 1, which applies to a more general region and more general position of the pole of the Green's function, follows directly by an inversion.

The Green's function for an arbitrary region R with pole at a point P has no critical points in any circular region which contains the point P but contains no point of the boundary of R.

By *circular region* we mean here the interior or exterior of a circle, or a half-plane.

This theorem may give fairly accurate knowledge of the location of critical points, when taken in conjunction with other known facts, such as that the set of critical points has whatever symmetry is possessed by the region R (taken together with P), and that the Green's function of a region of connectivity ρ has no more than $\rho-1$ distinct critical points. Thus, if R is a doubly connected region bounded by two circles C_1 and C_2, the unique critical point of the Green's function $G(x, y)$ for R with pole at P lies on the circle C through P orthogonal to C_1 and C_2. By the extension of Theorem 1, this critical point lies on that particular arc of C bounded by C_1 and C_2 which does not contain P.

6. *Analog of Jensen's Theorem.* The theorem of Jensen is well known:*

Let $p(z)$ be an arbitrary real polynomial (that is, with real coefficients) and let circles be drawn having as diameters the line segments joining the conjugate imaginary roots of $f(z)$. Then all the non-real roots of the derivative $p'(z)$ lie on or within these circles.

* Jensen, Acta Mathematica, vol. 36 (1912), p. 190. Walsh, Annals of Mathematics, vol. 22 (1920), pp. 128–144.

We shall prove the following analog, which also can be readily formulated for a more general region and a more general position of the pole.

THEOREM 2. *Let $G(x, y)$ be the Green's function with pole at infinity for a region R whose boundary B is finite and symmetric in a line L. Then all critical points of $G(x, y)$ not on L lie on or within the circles whose diameters are segments joining pairs of points of B symmetric with respect to L.*

It is to be noticed that these circles (which we shall call *Jensen circles*) are not merely circles whose centers lie on L and which contain all points of the boundary of R. It is essential that the circles have the segments indicated as *diameters*. The following remark is, however, useful.*

If the circle $x^2+y^2=r^2$ has on or within it a number of points symmetric in the axis of reals, then the ellipse $x^2+2y^2=2r^2$ has on or within it all the corresponding Jensen circles with respect to the axis of reals.

In order to prove Theorem 2, it is sufficient to prove that all critical points of $G(x, y)$ lie on or within a configuration K found by replacing each Jensen circle for B of radius r by a concentric circle whose radius is $r+\epsilon$, and adjoining the region $-\epsilon \leqq y \leqq \epsilon$, where $\epsilon > 0$ is arbitrary. Choose the lemniscates L_n symmetric in L. For suitably large n, all the Jensen circles for L_n lie interior to K. Indeed, a Jensen circle which corresponds to two points a_n and b_n of L_n symmetric in L such that $|a-a_n| < \epsilon/2$, $|b-b_n| < \epsilon/2$, where a and b belong to B and are symmetric in L, lies interior to K. Theorem 2 now follows by the method of §5. The following theorem reduces to Theorem 2 by an inversion.

Let $G(x, y)$ be the Green's function with pole at P for a region R whose boundary B is symmetric (anallagmatic) in a circle C which passes through P. Then every critical point of $G(x, y)$ not on C lies on a circle Q or is separated from P by a circle Q. A circle Q is any circle passing through a pair of points of B mutually inverse in C and orthogonal to the circle through P and that pair of points.

We remark, as a complement to Theorem 2 and under the hypothesis of that theorem, that *any open interval of L exterior*

* The introduction of these ellipses is again due to Jensen (loc. cit.). Formal proof of the remark is not difficult.

to all the Jensen circles and containing no point of B contains at most one critical point of $G(x, y)$; any open interval of L exterior to all the Jensen circles and containing no point of B but bounded by points of B contains precisely one critical point of $G(x, y)$. The former remark follows directly from the corresponding fact (Walsh, loc. cit.) for the polynomials which define the approximating lemniscates L_n. The latter remark follows (if L is chosen as the axis of reals) from the fact that $G(x, y)$ vanishes at both ends of the interval in question, so $\partial G/\partial x$ vanishes at some interior point P of the interval; everywhere in R on L and hence at P we have by symmetry $\partial G/\partial y = 0$, so P is a critical point of $G(x, y)$. In the two remarks just proved, the critical point of $G(x, y)$, if existent, must be simple: $|\partial^2 G/\partial x^2| + |\partial^2 G/\partial x \partial y| \neq 0$.

The following remark, also a complement to Theorem 2, is still more general but lies somewhat deeper; the proof is omitted.

Let S be a closed segment of L neither of whose end points belongs to B or lies interior to a Jensen circle for B. Let J be the configuration consisting of S and the closed interiors of all Jensen circles which intersect S. If B consists of a finite number of components (mutually exclusive closed point sets whose complements are simply connected) and if J contains precisely N of these components, then J contains precisely $N-1$, N, or $N+1$ critical points of $G(x, y)$. If S is the segment $x_1 \leqq x \leqq x_2$, $y=0$, the number of critical points of $G(x, y)$ belonging to J is

$$\begin{array}{lll} N-1, & \textit{if} & \partial G(x_1, 0)/\partial x < 0,\ \partial G(x_2, 0)/\partial x > 0; \\ N, & \textit{if} & [\partial G(x_1, 0)/\partial x]\cdot[\partial G(x_2, 0)/\partial x] > 0; \\ N+1, & \textit{if} & \partial G(x_1, 0)/\partial x > 0,\ \partial G(x_2, 0)/\partial x < 0. \end{array}$$

In this remark, account must be taken of the multiplicities of the critical points.

7. *Analog of Walsh's Theorem.* Another theorem of interest on the roots of the derivative of a polynomial is*

Let the circle C_1: $|z-\alpha_1| = r_1$ contain on or within it m_1 roots of a polynomial $p(z)$ and let the circle C_2: $|z-\alpha_2| = r_2$ contain on or within it all the remaining roots, m_2 in number, of $p(z)$; it is immaterial whether C_1 and C_2 contain roots of $p(z)$ other than those specified. Then all roots of $p'(z)$ lie on or within C_1, C_2, and the circle

* Walsh, Transactions of this Society, vol. 22 (1921), pp. 101–116; Comptes Rendus du Congrès des Mathématiciens (Strasbourg 1920), pp. 349–352.

$$C: \left| z - \frac{m_1\alpha_2 + m_2\alpha_1}{m_1 + m_2} \right| = \frac{m_1 r_2 + m_2 r_1}{m_1 + m_2} .$$

If these circles are mutually exterior, they contain, respectively, m_1-1, m_2-1, 1 *roots of* $p'(z)$.

We shall first be concerned with the following special case.

Let $p(z)$ *be a real polynomial whose roots lie on or within two circles* $C_1: |z-\alpha| = r$ *and* $C_2: |z-\bar{\alpha}| = r$. *Then all roots of* $p'(z)$ *lie on or within* C_1, C_2, *and* $C: |z-(\alpha+\bar{\alpha})/2| = r$. *If we have* $|\alpha-\bar{\alpha}| \geqq 4r$, *then one root of* $p'(z)$ *is real (it lies on the projection of* C_1 *on the axis of reals) and the other roots lie on or within* C_1 *and* C_2.

As a matter of fact, the first part of this theorem does not follow immediately from the preceding one, for the real polynomial $p(z)$ may have an *odd* number of real roots, so that we cannot set $m_1 = m_2$. In that case it is sufficient to apply the former theorem to the polynomial $[p(z)]^2$.

The following theorem is the precise analog for critical points of the Green's function; the proof is similar to those already given and hence is omitted.

THEOREM 3. *Let* R *be an infinite region of the plane whose boundary* B *is finite and symmetric in the axis of reals, and let* $G(x, y)$ *be the Green's function for* R *with pole at infinity. If all points of* B *lie on or within the circles* $C_1: |z-\alpha| = r$ *and* $C_2: |z-\bar{\alpha}| = r$, *then all critical points of* $G(x, y)$ *lie on or within* C_1, C_2, *and* $C: |z-(\alpha+\bar{\alpha})/2| = r$. *If we have* $|\alpha-\bar{\alpha}| \geqq 4r$, *then one critical point of* $G(x, y)$ *is real (it lies on the projection of* C_1 *on the axis of reals) and the others lie on or within* C_1 *and* C_2.

8. *Extensions.* Even if the Green's function $G(x, y)$ for a given region R does not exist, it may occur that the functions $G_n(x, y)$ defined in §3 (the sequence $G_n(x, y)$ is monotonic increasing) approach a limit $G(x, y)$ in R not the infinite constant. This new function $G(x, y)$ is then called the *generalized* Green's function for R; it is independent of the particular choice of the lemniscates L_n. Theorems 1, 2, and 3, and their proofs, are clearly valid for this generalized Green's function.

Still other applications of the theorems we have mentioned exist. For instance the following theorems can be proved by the method of §7.

Let the roots of the polynomial $p(z)$ be symmetric in the origin and lie on or within the two circles $C_1: |z-\alpha| = r$ and C_2: $|z+\alpha| = r$. Then all roots of $p'(z)$ lie on or within C_1, C_2, and $C: |z| = r$. If we have $|\alpha| \geqq 2r$, then one root of $p'(z)$ lies at the origin and the other roots lie on or within C_1 and C_2.

Let R be an infinite region whose boundary B is finite and symmetric in the origin, and let $G(x, y)$ be the Green's function for R with pole at infinity. If all points of B lie on or within the circles $C_1: |z-\alpha| = r$ and $C_2: |z+\alpha| = r$, then all critical points of $G(x, y)$ lie on or within C_1, C_2, and $C: |z| = r$. If we have $|\alpha| \geqq 2r$, then one critical point of $G(x, y)$ lies at the origin and the others lie on or within C_1 and C_2.

Many other theorems on the roots of the derivative of a polynomial can also be applied in the present situation. Let us state one further application.*

Let R be an infinite region with finite boundary B and let B have three-fold symmetry about the origin (that is, let B be unchanged by a rotation of $120°$ about the origin). Let all points of B lie on or within the three circles $|z-\omega h| \leqq r$, where ω is a cube root of unity. Then all critical points (exterior to these three circles) of the Green's function for R with pole at infinity lie on or within the circle $|z| \leqq (r^2+hr)^{1/2}$. In particular if we have $h \geqq 3r$, then all these critical points (except for a double critical point at the origin) lie on or within the three circles $|z-\omega h| \leqq r$.

In the present note we have used results on the location of the roots of the derivative of a polynomial to establish results on the location of the critical points of the Green's function. The reciprocal process is also possible, for if a polynomial $p(z)$ is given and if $\mu > 0$ is chosen sufficiently small, the various branches of the lemniscate $|p(z)| = \mu$ lie near the roots of $p(z)$, and the roots of the derivative $p'(z)$ are the critical points of the Green's function for the exterior of this lemniscate.

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* See Walsh, Proceedings of the National Academy of Sciences, vol. 8 (1922), pp. 139–141.