

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

239. Professor A. A. Albert: *Normal division algebras over algebraic number fields not of finite degree.*

The author proves here that every normal division algebra of degree n over an algebraic number field not of finite degree is cyclic and has exponent n . This theorem was known only for algebraic number fields of finite degree. By a trivial example he also shows that there exist non-cyclic normal simple algebras over algebraic number fields not of finite degree (while this is known not to be true over fields of finite degree). He finally considers also the problem of the equivalence of normal division algebras, reducing the problem for the fields here considered to the already solved problem for the case of finite degree. (Received July 22, 1933.)

240. Professor E. T. Bell: *Exponential numbers.*

An examination of some faulty physical calculations suggested the following problem. Let the power series expansion of $f(x)$ and the MacLaurin expansion of $\exp f(x)$ be absolutely convergent and termwise differentiable. It is required to devise some readily applicable arithmetical check on the accuracy of the heavy algebra usually involved in carrying the MacLaurin expansion to more than ten terms. This is provided in the present paper by congruences which the coefficient of $x^n/n!$ ($n=1, 2, \dots$) in the MacLaurin expansion of $\exp [f(x)-f(0)]$ must satisfy. Applications of the criteria disclose errors in the tabulated expansions, some of which have been copied from one handbook into several later compilations. (Received July 31, 1933.)

241. Professor E. T. Bell: *Polynomial diophantine systems.*

The method of reciprocal arrays, developed by the author for purely multiplicative diophantine systems, is here extended to diophantine systems consisting of any number of polynomials, in any numbers of indeterminates, of any degrees. Such systems as are solvable by the method are *completely* solvable (*all* sets of integers satisfying the system are given explicitly in parametric form). The theory is first developed for an abstract commutative ring. Application is then made to diophantine polynomial systems in any algebraic number field with class number unity. (Received July 21, 1933.)

242. Professor H. R. Brahana: *On cubic congruences.*

Any two irreducible cubic congruences with coefficients in the same modu-

lar field (the modulus being a prime) are conjugate under a linear fractional transformation on the variable with coefficients in the same field. If $x^3 - x + \beta \equiv 0$ is irreducible, then $x^3 - x + \beta(4 - 27\beta^2)^{1/2} \equiv 0$ is also irreducible. If p is of the form $6k + 1$ and $x^3 + \beta \equiv 0$ is irreducible, then $x^3 - \alpha x + (27\beta^2 + c^3\alpha^3)/(27\beta c^3) \equiv 0$, α and c being any numbers from 1 to $p - 1$, is irreducible, and every irreducible cubic with coefficient of x^2 equal to zero can be written in that form with β the same for all of them. (Received July 10, 1933.)

243. Professor H. R. Brahana: *On the isomorphisms of an abelian group of type 1, 1.*

The operators of order a power of p in the group of isomorphisms of the abelian group H of order p^n and type 1, 1, \dots belong to conjugate sets each of which is completely characterized by a partition $n = n_1 + n_2 + \dots + n_s$. Each such operator determines a subgroup of the holomorph of H whose characteristics (order, class, order of the central, order of the commutator subgroup, and order of the cross-cut of the central and commutator subgroup) are immediately expressible in terms of the numbers n_i . (Received July 10, 1933.)

244. Professor H. R. Brahana: *On the metabelian groups which contain a given group as a maximal invariant abelian subgroup.*

This paper is the beginning of a classification of the abelian subgroups of order p^m and type 1, 1, \dots of the group of isomorphisms of the abelian group of order p^n and type 1, 1, \dots . We consider only those subgroups whose operators are of two types, namely those corresponding to the partitions $n = 2 + 1 + 1 + \dots + 1$ and $n = 2 + 2 + 1 + \dots + 1$. Of the groups which contain at least one operator of the second type there are 6 types of order p^2 and 17 types of order p^3 . (Received July 10, 1933.)

245. Dr. Rothwell Stephens: *Note on a problem of Fréchet.*

Fréchet has proposed the following problem: Characterize the most general space in which there exists a non-constant continuous function. Its solution has been given by Urysohn for the spaces of Hausdorff, and by Chittenden for topological space. However, in his generalization Chittenden used for his definition of a continuous function a neighborhood definition which is not entirely adequate for general topological spaces. The solution of the problem using the Sierpinski definition of a continuous transformation is given in the following theorem: A necessary and sufficient condition that a topological space admit a non-constant continuous function is that it contain a normal family of thoroughly open sets and that it contain no singular points. (Received July 18, 1933.)

246. Professor R. L. Wilder: *Concerning a problem of K. Borsuk.*

K. Borsuk has raised a question (*Fundamenta Mathematicae*, vol. 20, p. 285, problem 54) as to whether a subcontinuum C of E_n which cuts E_n , and which is ϵ -transformable into a set C' such that $C \cdot C' = 0$, is an $(n - 1)$ -manifold. Although, as shown by examples, the answer to this question is negative (even if C is a Jordan continuum), several positive results are obtained. Thus (we assume C compact throughout) the complement of C is just two domains

with C as common boundary, and if C is a Jordan continuum, both these domains are uniformly locally connected; in the latter case, if $n=2$, C is a simple closed curve, and if $n=3$ and $p^1(C)$ is finite, C is a closed 2-dimensional manifold. If a continuum C that cuts E_n is deformable without meeting itself, then its complement is two uniformly locally connected domains and for $n=2, 3$, C is a closed $(n-1)$ -manifold. One by-product of these investigations is a sharpening of a recent theorem of ours (this Bulletin, vol. 37, p. 525, Abstract No. 258): In order that a compact continuum C in E_3 should be a closed 2-dimensional manifold, it is necessary and sufficient that it be a common boundary of at least two uniformly locally connected domains for one of which there exists an $\epsilon > 0$ such that none of its 1-cycles of diameter $< \epsilon$ links C . (Received June 24, 1933.)

247. Mr. J. A. Clarkson: *On double Riemann-Stieltjes integrals.*

We consider two definitions of double Riemann-Stieltjes integrals: that of the "restricted" integral is due to Fréchet (Nouvelles Annales, (4), vol. 10 (1910); Transactions of this Society, vol. 16 (1915); that of the "unrestricted" integral has recently been employed by Schoenberg and others (Transactions and Bulletin of this Society, 1933). The existence of the unrestricted integral for a particular integrator and integrand clearly implies the existence of the restricted integral (as well as the equality of their values); we show that the converse is not true. Fréchet has shown (a) that a sufficient condition for the existence of the restricted integral for every continuous integrand is that the integrator be of bounded variation in the Vitali sense; and (b) that a sufficient condition for the existence of the restricted integral for every integrand which is the product of a continuous function of x and a like function of y is that the integrator be of bounded variation in the Fréchet sense. We prove the necessity of the condition in both (a) and (b). We observe that in case (a) the condition specified is also sufficient for the existence of the unrestricted integral, and prove that in case (b) the condition stated is not sufficient to insure the existence of the unrestricted integral. (Received August 10, 1933.)

248. Professor R. P. Agnew: *On summability of multiple sequences.*

The author extends to sequences of arbitrary multiplicity Theorem 1 of his paper *On summability of double sequences*. American Journal of Mathematics, vol. 54 (1932), pp. 648-656, and hence obtains extensions of results of C. R. Adams, F. Lösch, and himself on summability of double sequences. (Received September 6, 1933.)

249. Professor P. A. Caris: *Integral solutions of $(u^4 - v^4) \cdot (x^4 - y^4) = z^4 - t^4$.*

The diophantine equation $P^4 + kQ^4 = R^4 + kS^4$, when solvable for a given k , implies that k can be represented as the quotient of two differences of bi-quadrates. It is here proved that it can be solved when k is itself the difference of two bi-quadrates. Thence are deduced integral solutions of the equation $(u^4 - v^4)(x^4 - y^4) = z^4 - t^4$. A two-parameter solution is given by $u = a + b$, $v = a - b$, $x = a^4 + 18a^2b^2 + 13b^4$, $y = 3(a^2 - b^2)(a^2 + 3b^2)$, $z = 2(a + b)(5b^4 + 6ab^3$

$+6a^3b - a^4$, $t = 2(a - b)(a^4 + 6a^3b + 6ab^3 - 5b^4)$, where a, b are arbitrary integers. (Received September 2, 1933.)

250. Mr. W. R. Church: *Tables of irreducible polynomials for the first four prime moduli.*

The author gives tables listing irreducible polynomials $\phi(x)$, modulo p , together with their exponents, that is, the order of the element x in the finite field defined by $\phi(x)$, within the following limits: for $p=2$, up to and including degree 11; for $p=3$, up to and including degree 7; for $p=5$, up to and including degree 5; for $p=7$, up to and including degree 4. (Received September 7, 1933.)

251. Dr. J. M. Clarkson: *An involutorial line transformation.*

Given a quadric H , a plane π not tangent to H , and a point O on H but not on π . In the plane π , consider a Cremona involutorial transformation I_n of order n . An arbitrary line t meets H in two points A_1, A_2 . Project H from O upon π by the projection P . The points A_1, A_2 are carried into A_1', A_2' by the transformation PI_nP^{-1} . The line $t' \equiv A_1'A_2'$ shall be called the conjugate of t by the line transformation T ; t is carried into t' by the transformation T . The order, singular lines, and invariant lines of T are discussed. (Received September 5, 1933.)

252. Professor G. C. Evans: *Inferior limit of Newtonian potential at points of mass from points of empty space.*

Given in space of three dimensions a closed, bounded, reduced set t , which is the boundary of an unbounded region T and possibly also of other regions which constitute an open set B , and bears an arbitrary distribution of positive mass, finite in total amount, of which the Newtonian potential at M is $V(M)$, the problem deals with the validity of the equation (1): $\inf \lim (M=Q)V(M) = V(Q)$, Q in t , M in $T+B$. It is valid, if in the neighborhood of Q , t does not contain a collection of rectangular boundaries, with sides parallel to two fixed but arbitrary normal directions, whose vertices constitute a set of positive spatial measure. In particular, for the logarithmic potential, we have the topological situation that (1) holds if t is closed and bounded, and B is vacuous. But it is shown by an example that the corresponding topological theorem is not valid in three dimensions. The problem is related to various questions of approximation and unicity, in particular to the uniqueness of capacity distributions the upper bound of whose potential is unity. (Received August 26, 1933.)

253. Mr. F. B. Jones: *Concerning normal and completely normal spaces.*

The author shows that if a separable Fréchet space- L is normal, every point set of power c has a limit point, and if it is completely normal, every point set of power c contains one of its limit points. If $\aleph_1 = c$, there exists an example of a normal separable Fréchet space- L in which every uncountable point set has a limit point, but in which there exists an uncountable point set that contains none of its limit points. Furthermore, a separable normal space satisfying the first three parts of Axiom 1 of R. L. Moore's *Foundations of Point Set Theory* is completely separable and therefore metric, provided that in this

space every uncountable point set contains a subset of power c . Also, a normal space satisfying the first three parts of Axiom 1 is completely normal. (Received August 26, 1933.)

254. Mr. F. B. Jones: *Concerning R. L. Moore's Axiom 5₁*. Preliminary report.

Consider the following axioms. Axiom 5₁*: *If P is a point of a region R , there exists in R a domain D containing P whose boundary is a subset of the sum of a finite number of continua lying in $R - D$.* Axiom $\bar{5}_1$: *If A is a point of a region R and B is a point distinct from A , there exists in R a closed and compact point set M separating A from B .* The author shows that if S is a non-acyclic space satisfying Axiom 5₁* together with Axioms 0, 1, 2, and 4 of R. L. Moore's *Foundations of Point Set Theory* (American Mathematical Society Colloquium Publications, vol. 13, 1932) then (1) space is cyclically connected, (2) if C , the boundary of a connected domain D , is a compact continuous curve, then every point of C is accessible from D , (3) Theorems 10, 12, 14, 16, 18, 19, 21, 22, 23, 24, 25, and a number of the remaining theorems of Chapter IV of the above work hold true. Furthermore, from Axiom $\bar{5}_1$ together with Moore's Axioms 0, 1, 2, and 4 it follows that either space is acyclic or Moore's Axiom 5 holds true. (Received August 26, 1933.)

255. Professor R. L. Moore: *Concerning compact continua which contain no continuum that separates the plane.*

Let S denote a definite compact plane continuum containing no continuum that separates the plane. For every two distinct points X and Y of S , let XY denote the irreducible subcontinuum of S from X to Y . In this paper the following theorems are established: I. If A and B are points of S , then (a) there exists a point O of S such that AO contains AB but is not a proper subset of AZ for any point Z of S , (b) if XY contains a continuum AO satisfying these conditions, then either $XY = OX$ or $XY = OY$, (c) there exists a point C such that OC contains AB but is not a proper subset of any irreducible subcontinuum of S . II. In order that S should be triodic it is necessary and sufficient that there should exist three points of S such that no one of them separates the remaining two from each other in S in the weak sense. III. If S is non-degenerate and atriodic, it is irreducible between some two of its points. (Received September 5, 1933.)

256. Dr. A. F. Moursund: *On summation of derived series of the conjugate Fourier series.*

The principal result of this paper is a theorem concerning the summability of the r th, $r \geq 0$, derived series of the conjugate Fourier series. The author has been unable to find in the literature a single such theorem for the case $r > 1$. A limit $\bar{f}^{(r)}(x)$ involving generalized derivatives (in the sense of de la Vallée Poussin) of orders $r-1, r-3, \dots$, is defined; sufficient conditions for the existence of $\bar{f}^{(r)}(x)$ at a point and almost everywhere in an interval are given; and it is shown that the r th, $0 \leq r \leq p-1$, derived series of the conjugate Fourier series of a Lebesgue integrable function $f(x)$ is summable by the N_{2p} method (defined by the author in a paper to appear in the *Annals of Mathematics*) to $\bar{f}^{(r)}(x)$ wherever that limit exists. The N_{2p} method includes as a

special case the second form of the Bosanquet-Linfoot (α, β) method, $\alpha = p$, $\beta > 1$ or $\alpha > p$ (Quarterly Journal of Mathematics, Oxford Series, vol. 2 (1931), pp. 207-229), which when $\alpha = p + \delta$, $\delta > 0$, and $\beta = 0$ is the Riesz equivalent of the Cesàro method $(C, p + \delta)$, $\delta > 0$. (Received September 2, 1933.)

257. Dr. R. E. A. C. Paley and Professor Norbert Wiener: *Notes on the theory and application of Fourier transforms. V: On entire functions.*

The authors give an extension by general Tauberian methods of Titchmarsh's theory of the properties of entire functions of semi-exponential type with real negative zeros. (Received August 23, 1933.)

258. Dr. R. E. A. C. Paley and Professor Norbert Wiener: *Notes on the theory and application of Fourier transforms. VI: On two problems of Pólya.*

The authors answer and discuss two problems on entire functions set by Pólya in the Jahresbericht for 1931. (Received August 23, 1933.)

259. Dr. R. E. A. C. Paley and Professor Norbert Wiener: *Notes on the theory and application of Fourier transforms. VII: On the Volterra equation.*

By an extension to Fourier integrals of a theorem of Wiener concerning absolutely convergent Fourier series, the authors prove a generalization of Mercer's theorem concerning the equivalence of the convergence of a series and of an average between its ordinary and its Cesàro sum. (Received August 23, 1933.)

260. Dr. R. E. A. C. Paley and Professor Norbert Wiener: *Notes on the theory and application of Fourier transforms. VIII: On the closure of sets of complex exponential polynomials.*

The authors discuss the theory of the closure of non-harmonic sets of complex exponential functions. (Received August 23, 1933.)

261. Dr. R. E. A. C. Paley and Professor Norbert Wiener: *Notes on the theory and application of Fourier transforms. IX: On non-harmonic Fourier series.*

The authors show that if a non-harmonic Fourier series is nearly enough harmonic, in the sense that the maximum difference between n and the n th frequency is small enough, its convergence in the mean, convergence, and summability theories are substantially identical with those of ordinary Fourier series. (Received August 23, 1933.)

262. Dr. A. H. Smith: *On the summability of derived conjugate series of the Fourier-Lebesgue type.*

This note, which is an extension of the author's paper *On the summability of derived series of the Fourier-Lebesgue type* (Quarterly Journal of Mathematics, Oxford Series, vol. 4 (1933), pp. 93-106), applies the method of summation

introduced by Bosanquet and Linfoot (Journal of the London Mathematical Society, vol. 6 (1931), pp. 117-126) to the r th derived conjugate series of the Fourier series generated by a periodic function integrable in the sense of Lebesgue. (Received August 30, 1933.)

263. Dr. Hassler Whitney (National Research Fellow): *Differentiable functions defined in closed sets. I.*

Let A be a closed set on the x -axis E , and let $f(x)$ be defined in A . If $f(x)$ is "of class C^m in A ," its definition can be extended over E so that it will have a continuous m th derivative there (see this Bulletin, January, 1933, p. 31). If x_0, \dots, x_m are distinct points of A , and $P(x) = c_0 + \dots + c_m x^m$ is the polynomial of degree at most m such that $P(x_i) = f(x_i)$ ($i = 0, \dots, m$), define the difference quotient $\Delta(x_0, \dots, x_m) = m!c_m$. It is shown that a necessary and sufficient condition that $f(x)$ be of class C^m in A is that for each point x of A and every $\epsilon > 0$ there exist a $\delta > 0$ such that if x_0, \dots, x_m and x'_0, \dots, x'_m are any two sets of distinct points of A within δ of x , then $|\Delta(x'_0, \dots, x'_m) - \Delta(x_0, \dots, x_m)| < \epsilon$. Taylor's formula in finite form is discussed. (Received September 8, 1933.)

264. Dr. G. T. Whyburn: *Concerning separating points.*

A point p of a metric space M is a separating point of M provided $M - p$ is separated between some pair of points of the components of M containing p . A point which is a separating point of some open set in M is called a local separating point of M . In this paper the principal properties of these types of points previously known for continua are developed for locally compact metric spaces in general. The treatment shows conclusively that the notion of a local separating point is a true localization of that of a separating point of a closed set rather than that of a continuum, to which it was formerly restricted. Incidentally it is shown that if a closed set X separates a component C of a compact metric space K into two separated sets C_a and C_b , then X will also separate C_a and C_b in K if, and only if, the limit of no convergent sequence of components of K intersects both C_a and C_b . (Received September 5, 1933.)

265. Dr. G. T. Whyburn: *Concerning maximal sets.*

Let T be any monotone system of closed subsets of an arbitrary metric space M , i.e., any closed subset of a T -set is itself a T -set. It is proved in this paper that if N is any non-degenerate subset of M such that N is not disconnected by the removal of any T -set, then there exists a maximal subset $H(N)$ of M containing N and having this property. Furthermore the sets $H(N)$ are continua, and the common part of any two of them is a T -set. By taking particular systems T , a large number of different types of maximal sets are obtained, some of which are already known. For example, if T is the system of all closed sets of dimension $\leq n-2$, we obtain for the sets $H(N)$ the so-called " n -dimensional components" of M . Also it is shown that if T is the system of all closed sets which contain no essential n -dimensional complete cycle, the sets $H(N)$ have the property that if K is any subcontinuum of M such that every n -cycle in K is ~ 0 in K , then every n -cycle in $K \cdot H(N)$ is ~ 0 in $K \cdot H(N)$. (Received September 5, 1933.)