ON THE GENERALIZED VANDERMONDE DETERMINANT AND SYMMETRIC FUNCTIONS

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1. *Introduction.* E. R. Heineman has published* a direct method for expressing an arbitrary symmetric polynomial in n variables $a_1, a_2, \cdots, a_n$ in terms of the elementary symmetric functions $E_1, E_2, \cdots, E_n$ of these variables. His method was based on a combination of a formula for the general Vandermonde determinant developed by him, with a theorem of Muir.† In the present note we develop in a very elementary way a simple formula for the quotient of a general Vandermonde determinant by the simple alternant, in terms of the "homogeneous product sums" of weight s.‡ These homogeneous product sums can be expressed explicitly in terms of the elementary symmetric functions.§ The formula which is obtained gives, therefore, by the use of Muir's theorem, another straightforward method for the calculation of an arbitrary symmetric polynomial.

2. *A Theorem on Determinants.* It is an exercise in elementary algebra to prove that the determinants

$$\begin{vmatrix} 1 & -b-c & bc \\ 1 & -c-a & ca \\ 1 & -a-b & ab \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

are equal. If, instead of evaluating each of the determinants, we try to transform one into the other by elementary transformations, we are led to the following generalization of this simple fact.

THEOREM 1. *If $a_1, \cdots, a_n$ are arbitrary complex numbers and $p_{ki} = (-1)^k \sum^{(i)} a_1 \cdots a_k$, the sum to be extended over all products k at a time of $a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n$, then*

* E. R. Heineman, *Generalized Vandermonde determinants*, Transactions of this Society, vol. 31 (1929), p. 464.

† Muir, *Theory of Determinants*, vol. 4, p. 151; see also Muir and Metzler, *A Treatise on the Theory of Determinants*, p. 344.

‡ See, for example, MacMahon, *Combinatory Analysis*, vol. 1, p. 3.

§ For example, as in MacMahon, loc. cit., p. 4.

$$(1)\qquad \begin{vmatrix} 1 & p_{11} & \cdots & p_{n-1,1} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 1 & p_{1n} & \cdots & p_{n-1,n} \end{vmatrix} = \begin{vmatrix} 1 & a_1 & \cdots & a_1^{n-1} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 1 & a_n & \cdots & a_n^{n-1} \end{vmatrix}.$$

PROOF. The coefficients p_k of the equation of degree n, $x^n+p_1x^{n-1}+\cdots+p_n=0$, which has as roots the given numbers $a_1, a_2, \cdots, a_n$, are related to the numbers p_{ki} by the equation $p_k=p_{k,i}+a_ip_{k-1,i}$. If the equations obtained by writing this relation successively for $k, k-1, \cdots, 1$, be multiplied by $1, a_i, \cdots, a_i^{k-1}$, respectively, and the results are added, we obtain the formula

$$(2)\qquad p_{ki} = \sum_{j=0}^{k} a_i^j p_{k-j};$$

here it is to be understood that $p_0=p_{0i}=1$. This formula as derived is valid only for $0\leqq k<n$. But it is immediately evident that it holds also for $k\geqq n$, *provided we put* $p_{ki}=0$, *for* $k\geqq n$. This will be our agreement throughout; it is in accord with the customary convention $p_k=0$, for $k>n$. Equation (1) is an immediate consequence of (2); for if we add to the $(k+1)$th column of the Vandermonde determinant, on the right side of (1), columns $1, 2, \cdots, k-1, k$ multiplied, respectively, by p_k, $p_{k-1}, \cdots, p_2, p_1$, it reduces to the $(k+1)$th column of the determinant on the left.

3. *Transformation of the Determinants in* (1). For our further purpose it is important to inquire how the left side of (1) can be transformed into the right side; the answer is given by solving equations (2) for a_i^j in terms of p_{ki}. This can be conveniently done in the following indirect way.

(a) If h_j represents the homogeneous product sum of degree j (that is the sum of all symmetric polynomials of degree j), then

$$(3)\qquad \sum_{j=0}^{k} h_j p_{k-j} = 0,$$

for $k\geqq 0$. For, from the identity

$$\prod_{i=1}^{n}(1 - a_i x) = \sum_{i=0}^{n} p_i x^i,$$

since for $|x| \leqq |1/a_i|$ we have

$$\frac{1}{1 - a_i x} = \sum_{j=0}^{\infty} a_i^j x^j,$$

and hence

$$\frac{1}{\prod_{i=1}^{n} (1 - a_i x)} = \prod_{i=1}^{n} \sum_{j=0}^{\infty} a_i^j x^j = \sum_{j=0}^{\infty} h_j x^j,$$

it follows that, for $|x| \leqq |1/a_i|$,

$$\sum_{i=0}^{n} p_i x^i \sum_{j=0}^{\infty} h_j x^j \equiv 1, \qquad (i = 1, \cdots, n).$$

Comparison of the coefficients of x^k shows that (3) is valid for any $k > 0$, since $p_k = 0$ for $k > n$; moreover $h_0 = 1$.

(b) If we write (2) in the form

$$a_i^k = p_{ki} - \sum_{j=0}^{k-1} a_i^j p_{k-j}, \tag{4}$$

it is easily proved that

$$a_i^k = \sum_{j=0}^{k} h_{k-j} p_{ji}, \qquad (0 \leqq k). \tag{5}$$

For $k = 0$, this reduces to the identity $1 \equiv 1$; for $k = 1$, to $a_i = h_1 + p_{1i}$, which is immediately verifiable. Assuming now that (5) holds for $k = 0, 1, \cdots, m$, we find by use of (4) and (3) that

$$\begin{aligned} a_i^{m+1} &= p_{m+1,i} - \sum_{j=0}^{m} a_i^j p_{m+1-j} = p_{m+1,i} - \sum_{j=0}^{m} \sum_{l=0}^{j} h_{j-l} p_{li} p_{m+1-j} \\ &= p_{m+1,i} - \sum_{l=0}^{m} \sum_{j=l}^{m} h_{j-l} p_{li} p_{m+1-j} \\ &= p_{m+1,i} - \sum_{l=0}^{m} p_{li} \sum_{j=0}^{m-l} h_j p_{m+1-j-l} \\ &= p_{m+1,i} + \sum_{l=0}^{m} p_{li} h_{m+1-l} = \sum_{l=0}^{m+1} h_{m+1-l} p_{li}. \end{aligned}$$

This completes the proof of (5); it shows that by using the multipliers $h_0, h_1, \cdots, h_{n-1}$ the determinant on the right of (1) is obtainable by elementary transformations from the one on the left.

4. *Transformation into Vandermonde Type.* But (5) can also be used to transform the determinant on the left of (1) into a generalized Vandermonde determinant. We shall designate the determinant on the left of (1) by its first row, thus $|1, p_{11}, \cdots, p_{n-1,1}|$. And we shall use the symbol $(0, m_1, m_2, \cdots, m_{n-1})$ to designate the generalized Vandermonde determinant whose ith row consists of the elements $1, a_i^{m_1}, a_i^{m_2}, \cdots, a_i^{m_{n-1}}$; we shall suppose $m_1 < m_2 < \cdots < m_{n-1}$. In this notation, Theorem 1 takes the form $|1, p_{11}, \cdots, p_{n-1,1}| = (0, 1, 2, \cdots, n-1)$. But we see furthermore that

$$(6) \qquad h_{m_{n-1}-n+1}\,|1, p_{11}, \cdots, p_{n-1,1}| = (0, 1, 2, \cdots, n-2, m_{n-1})$$

for arbitrary m_{n-1}.* Since $\sum_{j=0}^{m_{n-1}} p_{ji} h_{m_{n-1}-j} = a_i^{m_{n-1}}$, the multipliers $h_{m_{n-1}}, h_{m_{n-1}-1}, \cdots, h_{m_{n-1}-n+1}$ applied to the successive columns on the left transform the last column into the column of m_{n-1}th powers; for (5) holds with the understanding that $p_{ji} = 0$ for $j \geqq n$. Repeated application of (5) completes the transformation.

As an intermediate step in this transformation we have therefore the equality†

$$(7) \quad h_{m_{n-1}-n+1}\,|1, p_{11}, \cdots, p_{n-1,1}| = |1, p_{11}, \cdots, p_{n-2,1} a_1^{m_{n-1}}|.$$

We seek now to determine multipliers $\alpha, \alpha_0, \alpha_1, \cdots, \alpha_{n-2}$ such that

$$\alpha a_i^{m_{n-1}} + \alpha_0 + \alpha_1 p_{1i} + \cdots + \alpha_{n-2} p_{n-2,i} = a_i^{m_{n-2}}.$$

By (5), this condition is equivalent to

$$\alpha \sum_{j=0}^{m_{n-1}} h_{m_{n-1}-j} p_{ji} + \sum_{j=0}^{n-2} \alpha_j p_{ji} - \sum_{j=0}^{m_{n-2}} h_{m_{n-2}-j} p_{ji} = 0,$$

or since $p_{ji} = 0$ for $j > n-1$, to

* It is to be understood that $h_j = 0$, for $j < 0$; (6) is obviously true for $m_{n-1} < n-1$.

† This formula holds also if $h_{m_{n-1}-n+1} = 0$, although it can then not be obtained as above, because in that case $a_i^{m_{n-1}}$ is a linear combination of $1, p_{1i}, \cdots, p_{n-2,i}$.

$$(\alpha h_{m_{n-1}-n+1} - h_{m_{n-2}-n+1})p_{n-1,i} + \sum_{j=0}^{n-2}(\alpha h_{m_{n-1}-j} + \alpha_j - h_{m_{n-2}-j})p_{ji} = 0.$$

These multipliers are therefore uniquely determined by the equations

$$\alpha h_{m_{n-1}-n+1} - h_{m_{n-2}-n+1} = 0, \quad \alpha h_{m_{n-1}-j} + \alpha_j - h_{m_{n-2}-j} = 0,$$

for $j=0, \cdots, n-2$, if $h_{m_{n-1}-n+1} \neq 0$. The only one of them in whose actual value we are interested is α_{n-2}; for it we find

$$\alpha_{n-2} = \frac{h_{m_{n-1}-n+1}h_{m_{n-2}-n+2} - h_{m_{n-1}-n+2}h_{m_{n-2}-n+1}}{h_{m_{n-1}-n+1}}.$$

We find therefore from (7)

$$\begin{vmatrix} h_{m_{n-1}-n+1} & h_{m_{n-2}-n+1} \\ h_{m_{n-1}-n+2} & h_{m_{n-2}-n+2} \end{vmatrix} \cdot \left| 1, p_{11}, \cdots, p_{n-1,1} \right| = \left| 1, p_{11}, \cdots, p_{n-3,1}, a_1^{m_{n-2}}, a_1^{m_{n-1}} \right|, \tag{8}$$

and hence by repeated application of (5):

$$\begin{vmatrix} h_{m_{n-1}-n+1} & h_{m_{n-2}-n+1} \\ h_{m_{n-1}-n+2} & h_{m_{n-2}-n+2} \end{vmatrix} \cdot \left| 1, p_{11}, \cdots, p_{n-1,1} \right| = (0, 1, 2, \cdots, n-3, m_{n-2}, m_{n-1}). \tag{9}$$

It is readily seen, as in the last footnote above, that (8), and hence (9), also holds if $h_{m_{n-1}-n+1}=0$, and if $h_{m_{n-1}-n+1}h_{m_{n-2}-n+2} - h_{m_{n-1}-n+2}h_{m_{n-2}-n+1}=0$. A straightforward induction now enables us to prove the general formula foreshadowed in (6) and (9). Nothing is involved except the determination of multipliers and the use of (5). Hence we shall merely state the result.

THEOREM 2. *The general Vandermonde determinant* $(0, 1, 2, \cdots, k, m_{k+1}, m_{k+2}, \cdots, m_{n-1})$ *for* $0 \leqq k \leqq n-1$ *is equal to the determinant* $\left|1, p_{11}, \cdots, p_{n-1,1}\right|$ *multiplied by the determinant*

$$\begin{vmatrix} h_{m_{n-1}-n+1}, & h_{m_{n-2}-n+1}, & \cdots, & h_{m_{k+1}-n+1} \\ \cdot & \cdot & \cdot & \cdot \\ h_{m_{n-1}-k-1}, & h_{m_{n-2}-k-1}, & \cdots, & h_{m_{k+1}-k-1} \end{vmatrix},$$

where h_j *represents the homogeneous product sum of degree* j.*

* A proof of a theorem closely akin to this theorem and of several special cases of it, by methods quite different from those used above, is found in Muir and Metzler, loc. cit., pp. 329–336.

While $m_{k+1}, \cdots, m_{n-1}$ are arbitrary positive integers, there is obviously only interest in the case in which they are distinct positive integers, and $m_{k+1} > k+1$; in all other cases the corresponding determinant vanishes.

5. *Application of Muir's Theorem.* We bring in now the theorem of Muir referred to in the introduction, according to which the product of an arbitrary symmetric polynomial in n variables by the Vandermonde determinant $(0, 1, 2, \cdots, n-1)$ can be expressed as a sum of generalized Vandermonde determinants of the type here considered. In combination with Theorems 1 and 2, we obtain then the following result.

THEOREM 3. *The arbitrary symmetric polynomial in n variables* $\sum a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n}$, *where* α_i *are positive integers or zero, is equal to the sum of all determinants of the form*

$$\begin{vmatrix} h_{m_1-n+1}, & h_{m_2-n+1}, & \cdots, & h_{m_{n-1}-n+1} \\ \cdot & \cdot & \cdot & \cdot \\ h_{m_1-1}, & h_{m_2-1}, & \cdots, & h_{m_{n-1}-1} \end{vmatrix},$$

for which the sets of indices $m_1, m_2, \cdots, m_{n-1}$ *are obtained from the distinct permutations of the exponents* $\alpha_1, \alpha_2, \cdots, \alpha_n$ *by increasing their successive elements by* $0, 1, 2, \cdots, n-1$ *and then dropping the first element provided it is* 0. *If this first element is different from* 0, *the set* $m_1, m_2, \cdots, m_{n-1}$ *is obtained by permuting the modified exponents until the least among them, r, is in first place, then diminishing all of them by r and dropping the first one; in this case the corresponding determinant is to be multiplied by* $(-1)^\rho (a_1, a_2, \cdots, a_n)^r$, *where* ρ *is the number of transpositions required to bring the least exponent in first place.*

PROOF. By Muir's theorem, the product of $\sum a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n}$ by $(0, 1, 2, \cdots, n-1)$ is equal to the sum of all determinants of the form $(\alpha_1^{(i)}, \alpha_2^{(i)}+1, \cdots, \alpha_n^{(i)}+n-1)$, where the numbers $\alpha_1^{(i)}, \alpha_2^{(i)}, \cdots, \alpha_n^{(i)}$ range over the distinct permutations of $\alpha_1, \alpha_2, \cdots, \alpha_n$. If r is the least of all of the numbers $\alpha_1^{(i)}, \cdots, \alpha_n^{(i)}+n-1$ and if ρ transpositions bring it in first place, then $(\alpha_1^{(i)}, \cdots, \alpha_n^{(i)}+n-1) = (-1)^\rho (a_1, a_2, \cdots, a_n)^r \cdot (0, m_1, \cdots, m_{n-1})$, where $r, m_1+r, \cdots, m_{n-1}+r$ is a permutation of $\alpha_1^{(i)}, \cdots, \alpha_n^{(i)}+n-1$. Theorem 3 follows now mmediately from Theorems 1 and 2.

To express the homogeneous product sums in terms of elementary symmetric functions one can use the formula given by MacMahon and referred to in the introduction, the recursion formula (3), or the following explicit formula readily derivable from (3):

$$h_j = - \begin{vmatrix} 1, & 0, & \cdots, 0, & p_1 \\ p_1, & 1, & \cdots, 0, & p_2 \\ p_2, & p_1, & \cdots, 0, & p_3 \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ p_{j-1}, & p_{j-2}, & \cdots, p_1, & p_j \end{vmatrix}.$$

6. *Example.* To determine $\sum a^4b^3$ in 5 variables, that is, $\sum a^4b^3c^0d^0e^0$, we have $\alpha_1=4$, $\alpha_2=3$, $\alpha_3=\alpha_4=\alpha_5=0$. Hence there are 20 distinct permutations. Of these, the ones which contain any of the sequences 4 3, 4 x x x 0, or 3 x x 0 give rise to sets in which at least two elements are equal. There are left 9 permutations which lead to the following sets when their elements are increased by 0 1 2 3 4: (4 1 2 3 7), (0 5 2 6 4), (0 5 2 3 7), (0 1 6 3 7), (3 1 2 7 4), (0 1 5 7 4), (0 4 2 3 8), (0 1 5 3 8), and (0 1 2 6 8). Of these sets the first and fifth give rise to the same result, namely, to $-a_1\, a_2\, a_3\, a_4\, a_5\, h_2$; the remaining sets each lead to a 4th-order determinant. Development of these determinants gives

$$\sum a^4b^3c^0d^0e^0 = 2p_5h_2 - h_1^3\,h_4 + 3h_1^2\,h_2h_3 + 3h_1^2\,h_5 - h_1h_2^3$$
$$- 2h_1h_2h_4 - 5h_1h_3^2 - h_1h_6 + h_2^2\,h_3 - 3h_2h_5 + 7h_3h_4 - h_7.$$

Reduction to the p's leads, since $p_6=p_7=0$, to the result

$$\sum a^4b^3c^0d^0e^0 = -3p_1^3p_4 + 3p_1^2\,p_2p_3 + 7p_1^2\,p_5 - p_1p_2^3$$
$$+ 2p_1p_2p_4 - 5p_1p_3^2 + p_2^2\,p_3 - 7p_2p_5 + 5p_3p_4.$$

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