INVOLUTORIAL LINE TRANSFORMATIONS DETERMINED BY CREMONA PLANE INVOLUTIONS*

BY J. M. CLARKSON

1. *Introduction.* The author has discussed† an involutorial line transformation effected by considering a harmonic homology in each of two planes. If A, B be the points in which an arbitrary line (y) meets the planes α, β, and if A', B' be their images by the homologies I_α, I_β, respectively, then $(x)\equiv A'B'$ is the transform of (y). It is the purpose of the present paper to consider the line transformations similarly determined by Cremona involutorial transformations in each of two planes. All combinations of the four fundamental types: Homology; Jonquières; Geiser; and Bertini will be considered. The orders of the transformations, the invariant loci, the singular elements and the transforms of certain elementary forms are discussed.

2. *Homology-Jonquières.* In the plane α consider a harmonic homology I_α, center at O_1 and axis Δ_α. In the plane β consider the perspective Jonquières involution I_β, of order n, with basis point P_1 of multiplicity $(n-1)$ and basis points $P_2, \cdots, P_{2n-1}$ each simple, and with invariant curve $\Delta_\beta : P_1^{n-2}\ P_2^1 \cdots P_{2n-1}^1$ of order n and genus $(n-2)$.

An arbitrary line (y) meets α in a point A whose coordinates are linear in the Plücker coordinates y_i of (y) and meets β in a point B whose coordinates are also linear in y_i. The image A' of A by I_α has coordinates also linear in y_i but the image B' of B by I_β has coordinates which are functions of degree n in y_i. Hence $(x)\equiv A'B'$ has Plücker coordinates of degree $(n+1)$ in y_i. Thus, the transformation

$$x_i = \phi_i(y) \tag{1}$$

is of order $(n+1)$. The invariant lines of (1) form a congruence (n, n) composed of the lines meeting Δ_α, Δ_β; and in addition there is a cone of order n, vertex O_1, base curve Δ_β.

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† *Some involutorial line transformations*, this Bulletin, vol. 39 (1933), pp. 149–154.

If (y) meets β in P_1 and α in A, then $(y)\sim$ a cone of order $(n-1)$, vertex A', base curve the curve of order $(n-1)$ into which P_1 is transformed by I_β. If (y) meets β in $P_i(i\neq 1)$ and α in A, then $(y)\sim$ the pencil whose vertex is A' and in the plane $A'P_1P_i$.

The image by I_α of the line $c\equiv\alpha\beta$ is a line c_α and by I_β is a curve c_β of order n. The points C_α of c_α and C_β of c_β are projective with the points C of c. Hence the line joining any two corresponding points C_α, C_β is transformed by (1) into the bundle whose vertex is C. These lines $C_\alpha C_\beta$ form a regulus $\{c\}^{n+1}$ of order $(n+1)$, every generator of which is singular.

A line t_α in α meets β in a point C whose image by I_β is C_β. The image of t_α by I_α is a line t_α'. Hence $(y)\equiv t_\alpha\sim$ a pencil, vertex C_β, plane $C_\beta t_\alpha'$. If t_α pass through O_1, then $t_\alpha'\equiv t_\alpha$ and the plane of the pencil is $C_\beta t_\alpha$. If t_α be the axis Δ_α, then $t_\alpha'\equiv\Delta_\alpha$ and the plane of the pencil is $C_\beta\Delta_\alpha$.

A line t_β in β meets α in a point C whose image by I_α is C_α. The image of t_β by I_β is a curve ρ_β of order n. Hence $(y)\equiv t_\beta\sim$ a cone of order n, vertex C_α, base curve ρ_β. If t_β pass through P_1, then ρ_β is a line, and indeed the line t_β. Hence the conjugate is no longer a cone but a pencil, vertex C_α, plane $C_\alpha t_\beta$. If t_β pass through $P_i(i\neq 1)$, then ρ_β is of order $(n-1)$ and the conjugate cone is of order $(n-1)$. If t_β be the line P_1P_i, then $(y)\equiv t_\beta\sim C_\alpha P_i$.

As t_α describes the pencil (C, α) each conjugate pencil has its vertex at C_β and its plane passes through C_α. As C describes c, C_β describes c_β, and hence the plane field $(\alpha)\sim$ the special complex of order n with c_β as directrix curve.

As t_β describes the pencil (C, β), then each conjugate cone has its vertex at C_α and has $C_\alpha C_\beta$ as a generator. As C describes c, C_α describes c_α, and hence the plane field $(\beta)\sim$ the special linear complex $|c_\alpha|$.

Each line t_α of a pencil (T, α) is transformed by (1) into a pencil whose vertex is on c_β and whose plane passes through t_α' which belongs to the pencil (T', α), T' being the image by I_α of T. These ∞^1 pencils form a congruence $(n+1, n)$. If a line (x) meet c_β in a point C_β and also meet the t_α' which corresponds to C_β, then (x) belongs to the conjugate congruence. Through an arbitrary point of space a line (x) belonging to the conjugate congruence has coordinates which are of degree $(n+1)$ in the parameter λ of a line of (T, α). Hence the order of the congruence is $(n+1)$. An arbitrary plane meets c_β in n points and meets each corresponding t'_α in one point. Hence the class is n.

Likewise a pencil $(T, \beta) \sim$ a congruence $(n+1, n)$. However, if T lie at P_1, the order and class are both reduced so that the congruence is (2, 1). If T lie at $P_i (i \neq 1)$, then the congruence is $(n, n-1)$.

An arbitrary pencil $(T, \tau) \sim$ a regulus R of order $(n+1)$, the generators of which are the joins of corresponding points on the straight line image by I_α of $\tau\alpha$ and the curve of order n, image by I_β of $\tau\beta$. If τ pass through P_1, then R is of order 2; through $P_i (i \neq 1)$, of order n; through $P_iP_j (i, j \neq 1)$, of order $(n-1)$; through P_1P_i, the conjugate is a pencil with vertex P_i.

An arbitrary plane field of lines $(\tau) \sim$ a congruence (n, n) composed of lines meeting a line and a plane curve of order n not meeting the line. If τ pass through P_1, the congruence is (1, 1); through $P_i (i \neq 1)$, $(n-1, n-1)$; through $P_iP_j (i, j \neq 1)$, $(n-2, n-2)$; through P_1P_i, the conjugate is no longer a congruence but a pencil, vertex P_i.

An arbitrary bundle $(T) \sim$ a congruence $(3n, n)$. From an arbitrary point of space, the points of the planes α, β form two projective fields. There are three coincidences in a section of such a projection. Hence the parameters λ, μ of a line of (T) appear to degree $3n$ in defining a line (x) of the conjugate congruence through an arbitrary point of space. In an arbitrary plane of space lie n lines of the congruence.

A bilinear congruence $[|d_1|, |d_2|] \sim$ a congruence $(4n, 2n)$. The transformation (1) is involutorial. Hence the conjugate congruence of the (1, 1) will be of order equal to the number of lines common to the (1, 1) and the conjugate of an arbitrary bundle, which is $4n$. Likewise, the class will be the number of lines common to the (1, 1) and the conjugate of an arbitrary plane field.

A linear complex is transformed by (1) into a complex of order $(n+1)$ since this is the order of the transformation.

3. *Homology-Geiser.* Consider I_α in α as before, and in β, $I_\beta : P_1^3 \cdots P_7^3$, with invariant curve Δ_β of order 6 having double-points at $P_1, \cdots, P_7$. The order of the transformation is 9; the invariant lines form a congruence (6, 6) and a cone of order 6, vertex O_1, base curve Δ_β; the singular elements are, as

before, the lines of the bundles whose vertices are F-points of I_β, a regulus $\{c\}^9$ of order 9 and the plane fields (α), (β). To any line of any bundle corresponds a cone of order 3; to any generator of $\{c\}^9$, a bundle, vertex on c; to (α) a special complex of order 8, and to (β) a special linear complex. If t_β pass through one or two points P_i, its conjugate cone is of order 5 or 2. If the vertex T of a pencil (T, β) be at P_i, the conjugate congruence of the pencil is (6, 5); otherwise (T, α) or $(T, \beta) \sim$ a congruence (9, 8).

If the plane τ of an arbitrary pencil (T, τ) pass through one or two points P_i, the conjugate regulus is of order 6 or 3. Otherwise $(T, \tau) \sim$ a regulus of order 9.

If the plane τ pass through j points $P_i (j=0, 1, 2)$, $(\tau) \sim$ a congruence $(8-3j, 8-3j)$.

An arbitrary bundle $(T) \sim$ a congruence (24, 8).

A congruence (1, 1) $\sim$ a congruence (32, 16), and a linear complex $\sim$ a complex of order 9.

4. *Homology-Bertini*. Consider I_α as before and $I_\beta : P_1^6 \cdots P_8^6$, Δ_β being of order 9 with triple-points at P_i. The order of the transformation is 18; the invariant lines form a congruence (9, 9) and a cone of order 9; the singular elements are the lines of the bundles (P_i), a regulus $\{c\}^{18}$ of order 18, and the plane fields (α), (β). Any line of any $(P_i) \sim$ a cone of order 6, each generator of $\{c\}^{18} \sim$ a bundle with vertex on c, $(\alpha) \sim$ a special complex of order 17, and $(\beta) \sim$ a special linear complex. The conjugate cone of a line t_β has its order $17-6j$, where j is the number of points P_i on t_β. The conjugate congruence of a pencil (T, α) or (T, β) is (18, 17) unless T lie at some P_i, when the congruence is (12, 11).

If τ pass through j points $P_i (j=0, 1, 2)$, then the pencil $(T, \tau) \sim$ a regulus of order $18-6j$ and the plane field $(\tau) \sim$ a congruence $(17-6j, 17-6j)$.

A bundle $(T) \sim$ a congruence (51, 17).

A congruence (1, 1) $\sim$ a congruence (68, 34), and the transform of a linear complex is a complex of order 18.

5. *Jonquières-Jonquières*. When we consider two perspective Jonquières involutions, I_α of order m, center O_1, and I_β of order n, center P_1, the order of the transformation is $(m+n)$. The invariant lines form a congruence (mn, mn) since now Δ_α and Δ_β

are of orders m, n, respectively, and each passes simply through the simple F-points in its respective plane and multiply through the center. In addition to the singular regulus $\{c\}^{m+n}$ of order $(m+n)$ whose generators are transformed into bundles with vertices on c, and the singular plane fields (α), (β) whose conjugates are special complexes of orders n, m, respectively, and the singular bundles whose vertices are at F-points and each of whose lines is transformed into a cone whose order is the multiplicity of the F-point in I_α or I_β, there are $(2m-1)(2n-1)$ singular lines O_iP_j whose conjugates are congruences. If $i, j\neq 1$, each congruence is $(1, 1)$; if $i=1$, $j\neq 1$, each congruence is (m, m); if $i\neq 1$, $j=1$, each is (n, n); if $i=j=1$, the conjugate congruence is $([m-1][n-1], [m-1][n-1])$. A line $(y)\equiv t_\alpha\sim$ a cone of order $(n-j)$, where j is the sum of the multiplicities of F-points O_i on t_α, and $(y)\equiv t_\beta\sim$ a cone of order $(m-i)$, where i is the sum of the multiplicities of F-points P_l on t_β. A pencil $(T, \beta)\sim$ a congruence $(m+n-i, m[n-i])$ where i is the multiplicity of T as an F-point of I_β and $(T, \alpha)\sim$ a congruence $(m-j+n, [m-j]n)$, where j is the multiplicity of T as an F-point of I_α.

An arbitrary pencil $(T,\tau)\sim$ a regulus R of order $(m-i+n-j)$, where i is the sum of the multiplicities of F-points O_k and j the sum of the multiplicities of F-points P_l lying in τ. If $i=m$ or $j=n$, then R is a cone. Both $i=m$ and $j=n$ will not occur if I_α, I_β be taken arbitrarily.

A plane field $(\tau)\sim$ a congruence whose order and class are both $(m-i)(n-j)$, where i, j are as defined immediately above. In the event $m-i=0$ or $n-j=0$, the conjugate of (τ) is a cone whose order is the factor which does not vanish.

An arbitrary bundle $(T)\sim$ a congruence $(3mn, mn)$.

A congruence $(1, 1)\sim$ a congruence $(4mn, 2mn)$, and a linear complex has for conjugate a complex of order $(m+n)$.

6. *Jonquières-Geiser*. Take I_α as in §5 and I_β as in §3. The transformation is of order $(m+8)$; the invariant lines form a congruence $(6m, 6m)$; the singular elements are the bundles (O_i), (P_j), the $7(2m-1)$ lines O_iP_j, the plane fields (α), (β), and the singular regulus $\{c\}^{m+8}$ defined as in each previous case. Each line of $(O_i)\sim$ a cone of order $(m-k)$, where k is the multiplicity of O_i in I_α, and each line of $(P_j)\sim$ a cone of order 3;

$(y) \equiv O_iP_j \sim$ a congruence whose order and class are both $5(m-k)$, where k is as defined just above; $(\alpha) \sim$ a special complex of order 8 and $(\beta) \sim$ a special complex of order m; each generator of $\{c\}^{m+8} \sim$ a bundle whose vertex is on c. Finally $(y) \equiv t_\alpha \sim$ a cone of order $(m-i)$, where i is the sum of the multiplicities of F-points on t_α, and $(y) \equiv t_\beta \sim$ a cone of order $(8-j)$, where j is the sum of the multiplicities of F-points on t_β.

A pencil $(T, \alpha) \sim$ a congruence $(m-i+8, [m-i]8)$, where i is the multiplicity of T as an F-point of I_α, and a pencil $(T, \beta) \sim$ a congruence $(m+8-j, m[8-j])$, where j is the multiplicity of T as an F-point of I_β.

An arbitrary pencil $(T, \tau) \sim$ a regulus R whose order is $(m-i+8-j)$, where i is the sum of multiplicities of F-points O_k and j the sum of multiplicities of F-points P_l lying in τ. If I_α, I_β be taken arbitrarily, no more than 3 points O_k, P_l lie on τ. If $i=m$, R is a cone of order $(8-j)$.

A plane field $(\tau) \sim$ a congruence whose order and class are both $(m-i)(8-j)$, where i, j are as defined in the preceding paragraph. If $i=m$, the conjugate of (τ) is a cone of order $(8-j)$.

A bundle $(T) \sim$ a congruence $(24m, 8m)$.

A congruence $(1, 1) \sim$ a congruence $(32m, 16m)$ and a linear complex is transformed into complex of order $(m+8)$.

7. *Jonquières-Bertini.* Take I_α as in §5 and I_β as in §4. The transformation is of order $(m+17)$, and all of the results follow by replacing 8 by 17 in the preceding section, except that a bundle $(T) \sim$ a congruence $(51m, 17m)$ and a congruence $(1, 1) \sim$ a congruence $(68m, 34m)$.

8. *Geiser-Geiser.* Given $I_\alpha : O_1^3 \cdots O_7^3$ and I_β as in §3. The transformation is of order 16; the invariant lines form a congruence (36, 36); the singular elements are (O_i), (P_j), O_iP_j, $\{c\}^{16}$, (α), (β), and their conjugates are easily found by the methods outlined above. The conjugates of the elementary forms (T, τ), (τ), (T), $(1, 1)$ and linear complex are also readily obtained.

9. *Geiser-Bertini.* Consider I_α as in §8 and I_β as in §4. The transformation is of order 25; the invariant lines form a congruence (54, 54); the singular elements are (O_i), (P_j), O_iP_j, $\{c\}^{25}$, (α), (β), and their conjugates, and those of the elemen-

tary forms follow immediately from the method we have used throughout this paper.

10. *Bertini-Bertini.* Given $I_\alpha: O_1^6 \cdots O_8^6$ and I_β as in §4. The order of the transformation is 34, and by repetition of what has been done above we can discuss this transformation completely.

If fundamental elements of one or both Cremona plane involutions lie on the line $c \equiv \alpha\beta$ the preceding results must be modified. The details are not difficult in each particular case, but the large number of possible forms cannot be considered here.

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A SEPARATION THEOREM*

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Various writers on topology have had occasion in the course of their work to prove lemmas of the following general nature. Given sets A and B lying in a connected space Z, it is possible to express Z as the union of two continua M and N such that $N\cdot(A - A\cdot B) = M\cdot(B - A\cdot B) = 0$, provided that A, B, and Z satisfy the proper conditions. The last of these to come to the writer's attention are two theorems by Vietoris and one by the author of this note.† Such theorems are of course generalizations of Tietze's separation axioms‡ and it might prove profitable to work out systematically the possibilities along this line.

In some of the generalizations mentioned it is shown that, if Z is locally connected, a decomposition $Z = M + N$, where M and N are also locally connected, is possible, but, as far as the writer knows, the following theorem, which shows a certain kind of local connectivity for $M\cdot N$ as well as for M and N, is new.

THEOREM. *Let A and B be sub-continua of the locally connected compact metric space Z and let $A\cdot B$ be totally disconnected or void.*

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† L. Vietoris, *Über den höheren Zusammenhang*, Fundamenta Mathematicae, vol. 19, pp. 271–272; W. A. Wilson, *On unicoherency about a simple closed curve*, American Journal of Mathematics, vol. 55, p. 141.

‡ See F. Hausdorff, *Mengenlehre*, p. 229.