$$\int_{2\delta}^{a}\left|\log\left(1-\frac{\delta}{\tau}\right)\right|\frac{d\tau}{\tau} = -\int_{1/2}^{\delta/a}\left|\log(1-\tau)\right|\frac{d\tau}{\tau}$$

$$= \int_{\delta/a}^{1/2}\left|\log(1-\tau)\right|\frac{d\tau}{\tau} = o(1).$$

Hence $I_2 = o(1)$. Finally it is obvious that $I_3 = o(1)$ as $\delta \to 0$. On combining these facts, we obtain the desired result:

$$\int_{\delta}^{a}\left|\phi(t+\delta)-\phi(t)\right|\frac{dt}{t} = o(1).$$

Brown University

A NOTE ON COMPACTNESS*

BY E. H. HANSON

The purpose of this note is to deduce the conditions for compactness† of a set of measurable functions from the general criterion for compactness in complete metric spaces.‡ This is the procedure that suggests itself immediately and it suceeds without any difficulty. The general criterion referred to asserts that a set S of elements of a complete metric space is compact if and only if, for every positive ϵ, S is inclosable in a finite number of spheres§ of radius ϵ; and the validity of this the reader may easily verify for himself. Fréchet|| has obtained the result for

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† A set S of elements of a space is *compact* if every infinite subset of S has at least one limit point in the space.

‡ A space M is said to be *metric* if there exists a positive or zero function $d(e_1, e_2)$ of pairs of elements of M satisfying the conditions: (1) $d(e_1, e_2) = d(e_2, e_1)$, (2) $d(e_1, e_2) = 0$ is equivalent to $e_1 = e_2$; (3) $d(e_1, e_3) \leqq d(e_1, e_2) + d(e_2, e_3)$. A metric space is *complete* if $\lim_{m,n\to\infty} d(e_m, e_n) = 0$ implies the existence of an element e such that $\lim_{n\to\infty} d(e_n, e) = 0$.

§ A *sphere* with center c and radius r is by definition the set of elements e of M such that $d(e, c) < r$.

|| *Sur les ensembles compacts de fonctions mesurables*, Fundamenta Mathematicae, vol. 9, p. 25.

measurable functions but without making the deduction in the direct manner made possible by the use of the general criterion; the deduction given here, it would seem, is the simplest available.

To apply the above criterion to the case of measurable functions we must define the distance or écart of two measurable functions. The idea leading to this definition is that the distance between two measurable functions will be regarded as small if the functions differ by a small amount except in a set of small measure. The distance $d(f, \phi)$ of two measurable functions $f(x)$ and $\phi(x)$ defined on a bounded measurable set E will accordingly be defined as the lower boundary of all positive numbers ω such that $|f(x)-\phi(x)|<\omega$ except at the points of a set of measure $<\omega$. Convergence in terms of this distance is evidently equivalent to convergence in measure. That the set of all measurable functions defined on E constitutes a metric space follows at once if we agree to regard as identical two measurable functions which differ only on a set of measure zero. It is also easy to verify that this space is complete.* It follows that the above general criterion holds for this space. This is the result desired except that what is left to do is to mold it over into another form; and here the argument is at once suggested by the corresponding argument for continuous functions and by the fact that every measurable function is equal to a continuous function with an arbitrarily small error, except in a set of arbitrarily small measure. For continuous functions the resulting theorem is: A necessary and sufficient condition that a set of continuous functions defined on a bounded, closed set be compact† is that the set of functions be equi-bounded (uniformly bounded) and equi-continuous.‡

Let S be any set of measurable functions defined on a bounded measurable set E and inclosable, for every $\epsilon>0$, in a finite number of spheres of radius ϵ. For a given ϵ denote the spheres by

* Or we may note that this is merely a restatement of a well known property of convergence in measure. See Hobson, *Theory of Functions of a Real Variable*, vol. II, p. 244.

† For continuous functions $d(f_1, f_2) = \text{maximum } |f_1(x)-f_2(x)|$.

‡ A set of continuous functions is said to be *equi-continuous* if, for every positive ϵ, there exists a positive number δ such that $|x_1-x_2|<\delta$ implies for every function $f(x)$ of the set, that $|f(x_1)-f(x_2)|<\epsilon$.

H_i and their centers by $f_i(x)$, $[i=1, 2, \cdots, n]$. Since $f_\nu(x)$ is measurable it can be approximated by a continuous function $c_\nu(x)$ with an error $<\epsilon$ except in a set of E_ν of measure $<\epsilon$. Since $f_0(x)$ may be extended to a measurable function defined in a closed interval I containing E, and since $c_\nu(x)$ may be defined on I, there is no restriction in assuming that $c_\nu(x)$ is uniformly continuous on E. If $f(x)$ is in the sphere with center $f_\nu(x)$, we have $d(f_\nu, f)<\epsilon$ and since $d(c_\nu, f_\nu)<\epsilon$ we have $d(f, c_\nu)<2\epsilon$, that is, $f(x)$ differs from $c_\nu(x)$ by less than 2ϵ except on a set of measure $<2\epsilon$. Since the $c_i(x)$ are continuous and finite in number it follows that they are equi-bounded and equi-continuous. Therefore S is *almost equi-bounded* and *almost equi-continuous* in the sense that, for every $\epsilon>0$, there exist two positive numbers M and δ such that to every function $f(x)$ of S there corresponds a set E_f of measure $<\epsilon$ such that, except on E_f, $|f(x)|<M$ and such that, for every x_1, x_2 belonging to $E-E_f$, $|x_1-x_2|<\delta$ implies $|f(x_1)-f(x_2)|<\epsilon$.

What now remains is to verify the converse, namely, that, if S is any almost equi-bounded and almost equi-continuous set of measurable functions defined on a bounded measurable set E, then, for every $\epsilon>0$, S is inclosable in a finite number of spheres of radius ϵ. Let, then, ϵ be given. Since E lies in a finite interval and since S is almost equi-bounded and almost equi-continuous, it is possible to determine a rectangle R and a positive number δ such that with every function $f(x)$ of S there is associated a set E_f of measure $<\epsilon$ such that, on $E-E_f$, $f(x)$ lies entirely in the interior of R and such that, x_1, x_2 being any two points of $E-E_f$, $|x_1-x_2|<\delta$ implies $|f(x_1)-f(x_2)|<\epsilon$. Divide R into a finite number of vertical strips of width $<\delta$ by means of vertical lines having abscissas $x_0, x_1, \cdots, x_r$ and also into a finite number of horizontal strips of width $<\epsilon$ by lines having the ordinates $y_0, y_1, \cdots, y_s$. Let $f(x)$ now denote a particular function of S and consider the corresponding set $E-E_f$. The measure of this set is $>m(E)-\epsilon$. We may accordingly determine a closed subset F of $E-E_f$ which is also of measure $>m(E)-\epsilon$. We now define a measurable function $\phi(x)$ on the set E as follows. If there are no points of F in the interval $x_k\leqq x<x_{k+1}$, we define $\phi(x)=0$ for every point of E in this interval. If there are points of F in $x_k\leqq x<x_{k+1}$, let ξ be the first. The point $[\xi, f(\xi)]$ is in R. Suppose $y_l\leqq f(\xi)<y_{l+1}$. We define

$\phi(x) = y_l$ for every point of E in $x_k \leqq x < x_{k+1}$. Then, at every point of F, $|f(x) - \phi(x)| < \epsilon$, and $m(E - F) < \epsilon$. Hence $d(f, \phi) < \epsilon$. But the number of such functions $\phi(x)$ is finite, and S is inclosable in the above sense. We have shown that, for the space of measurable functions, the general criterion for compactness in complete metric spaces is equivalent to the following theorem.

THEOREM OF FRÉCHET. *A necessary and sufficient condition that a set S of measurable functions defined on a bounded measurable set E be compact is that the functions of S be almost equibounded and almost equi-continuous.*

The difficulty of generalizing the above results to general functions so as to obtain useful results is shown by an example due to Sierpiński* from which it is apparent that, if a distance or écart is to be defined satisfying certain intuitive ideas of distance and also satisfying the condition that ordinary convergence is to imply convergence in terms of the distance, then the general functions must be so classified that all of the functions $f(x) \equiv a$ will be regarded as identical.

THE OHIO STATE UNIVERSITY

* Fundamenta Mathematicae, vol. 9, p. 34.