ON THE NUMBER OF $(q+1)$-SECANT S_{q-1}'S OF A CERTAIN V_k^n IN AN $S_{qk+q+k-1}$

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In this note we are concerned only with those k-dimensional non-developable varieties which are rational loci each of ∞^1 $(k-1)$-spaces. By a rational locus of ∞^1 $(k-1)$-spaces we mean one whose $(k-1)$-spaces can be put in a one-to-one correspondence with the points of a straight line. Let such a locus or variety, V_k^n, of order n be given in an S_r. Now in S_r there are $\infty^{q(r-q+1)}$ $(q-1)$-spaces. For a $(q-1)$-space to meet V_k^n $q+1$ times is equivalent to $(q+1)(r-q-k+1)$ simple conditions. In order that the number, N, of $(q-1)$-spaces $(q+1)$-secant to V_k^n, that is, having $q+1$ points of simple incidence with V_k^n, be finite, we must have $(q+1)(r-q-k+1)=q(r-q+1)$ or $r=qk+q+k-1$. It is our purpose to determine this number N of $(q+1)$-secant S_{q-1}'s of V_k^n in $S_{qk+q+k-1}$.

For this purpose we find it convenient to consider the V_k^n in question as the projection of a $V_k'^n$ in a higher space $S_{r'}$. This $V_k'^n$ may always be regarded as the locus of ∞^1 $(k-1)$-spaces joining corresponding points of k rational, projectively related curves $C^{n_1}, C^{n_2}, \cdots, C^{n_k}$ of respective orders $n_1, n_2, \cdots, n_k$, where $n_1+n_2+\cdots+n_k=n$. The $S_{r'}$ containing $V_k'^n$ must be such that $r'\leqq n+k-1$. If $r'=n+k-1$, $V_k'^n$ is said to be normal in S_{n+k-1}. It is only necessary to consider this normal $V_k'^n$.

Let the k curves be given parametrically by

$$C^{n_1}\quad x_0:x_1:\cdots:x_{n_1}=t^{n_1}:t^{n_1-1}:\cdots:1,$$
$$x_{n_1+1}=x_{n_1+2}=\cdots=x_{n+k-1}=0;$$
$$C^{n_2}\quad x_0=x_1=\cdots=x_{n_1}=0,$$
$$x_{n_1+1}:x_{n_1+2}:\cdots:x_{n_1+n_2+1}=t^{n_2}:t^{n_2-1}:\cdots:1,$$
$$x_{n_1+n_2+2}=x_{n_1+n_2+3}=\cdots=x_{n+k-1}=0;$$

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$$C^{n_k}\quad x_0=x_1=\cdots=x_{n-n_k+k-2}=0,$$
$$x_{n-n_k+k-1}:x_{n-n_k+k}:\cdots:x_{n+k-1}=t^{n_k}:t^{n_k-1}:\cdots:1.$$

Then a general point of $V_k'^n$ has the coordinates

$$(\lambda_1 t^{n_1}:\lambda_1 t^{n_1-1}: \cdots :\lambda_1:\lambda_2 t^{n_2}:\lambda_2 t^{n_2-1}: \cdots :\lambda_2: \cdots : \cdots :\lambda_k t^{n_k}:\lambda_k t^{n_k-1}: \cdots :\lambda_k).$$

Now let t take on $q+1$ values, say $t_0, t_1, \cdots, t_q$, and we have $q+1$ points on $V_k'^n$ determining an S_q. The parametric equations of this S_q are, the parameters being the l's,

$$x_{n-n_h+h-1+j_h} = \lambda_h \sum_{i=0}^{q} (l_i t_i^{n_h-j_h}),$$

$$[h = 1, 2, \cdots, k; j_h = 1, 2, \cdots, n_h].$$

If we now eliminate the t's, l's, and λ's from the above equations of S_q, we obtain

$$\left\| \begin{matrix} x_0 & x_1 & \cdots & x_{n_1-q-1} & x_{n_1+1} & x_{n_1+2} & \cdots & x_{n_1+n_2-q} & \cdots & \cdots & x_{n-n_k+k-1} & x_{n-n_k+k} & \cdots & x_{n+k-q-2} \\ x_1 & x_2 & \cdots & x_{n_1-q} & x_{n_1+2} & x_{n_1+3} & \cdots & x_{n_1+n_2-q+1} & \cdots & \cdots & x_{n-n_k+k} & x_{n-n_k+k+1} & \cdots & x_{n+k-q-1} \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & & & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & & & \cdot & \cdot & & \cdot \\ x_{q+1} & x_{q+2} & \cdots & x_{n_1} & x_{n_1+q+2} & x_{n_1+q+3} & \cdots & x_{n_1+n_2+1} & \cdots & \cdots & x_{n-n_k+k+q} & x_{n-n_k+k+q+1} & \cdots & x_{n+k-1} \end{matrix} \right\| = 0.$$

These are the equations of a $(qk+q+k)$-dimensional variety V_{qk+q+k}^M of order M. This variety is the locus of the $\infty^{k(q+1)}$ q-spaces each meeting $V_k'^n$ $q+1$ times. To determine M, notice that the matrix in the left-hand member of the above equality consists of $n-qk$ columns and $q+2$ rows. Applying the rule given by Salmon* for the determination of the order of a restricted system of equations, we find that the order of V_{qk+q+k}^M is

$$M = \binom{n-qk}{q+1}.$$

Since V_{qk+q+k}^M is in S_{n+k-1}, an $S_{n-qk-q-1}$ of S_{n+k-1} meets it in

* *Modern Higher Algebra*, 4th ed., Lesson 19.

M points. Now let both $V_k'^n$ and V_{qk+q+k}^M be projected from $S_{n-qk-q-1}$ upon an S_{qk+q+k}. The projection of the former is a $V_k''^n$ and that of the latter is a system of $\infty^{k(q+1)}$ q-spaces. Each of these q-spaces is $(q+1)$-secant to $V_k''^n$ and M of them pass through a given point P. If we now project $V_k''^n$ from P upon an $S_{qk+q+k-1}$ of S_{qk+q+k}, we obtain for projection the V_k^n the number N of whose $(q+1)$-secant $(q-1)$-spaces we wish to find. The $(q+1)$-secant S_{q-1}'s of V_k^n are the $(q-1)$-spaces in which $S_{qk+q+k-1}$ intersects the $(q+1)$-secant S_q's of $V_k''^n$ passing through P. Hence the number N we are seeking is equal to M, that is,

$$N = \binom{n-qk}{q+1}.$$

Thus, for $k=1$, we have a rational curve C^n in S_{2q} having $\binom{n-q}{q+1}$ $(q+1)$-secant S_{q-1}'s. If $q=1$, we have the familiar case of a rational plane curve of order n with $(n-1)(n-2)/2$ double points. If $q=2$, we have the case which is also familiar of a rational 4-space curve having $(n-2)(n-3)(n-4)/6$ trisecant lines.

Let $k=2$ and we have a rational ruled surface F^n of order n in S_{3q+1} with $\binom{n-2q}{q+1}$ $(q+1)$-secant S_{q-1}'s. Thus, a rational F^n in S_4 has $(n-2)(n-3)/2$ improper double points; an F^n in S_7 has $(n-4)(n-5)(n-6)/6$ trisecant lines.

If we put $k=3$ and then $q=1, 2, 3, \cdots$, successively, we find, by what precedes, that a rational planed variety V_3^n of order n in S_6, S_{10}, $S_{14}, \cdots$, has, respectively, $(n-3)$ $(n-4)/2$ improper double points, $(n-6)(n-7)(n-8)/6$ trisecant lines, $(n-9)(n-10)(n-11)(n-12)/24$ quadrisecant planes, $\cdots$.

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