given probability w, with $0<w<1$, a Bernoulli sequence can be constructed; in fact, for the same w, a set of sequences having the power of the continuum. The "law of large numbers" and the "fundamental law of probability" leading to the "normal" probability function and its integral are then established. The Lexis dispersion theory concludes the text proper. Then follow: a one-page 4-place probability integral table; answers to the 18 exercises given in the text; a short list of important books and papers; an index for authors and subjects.

Kamke's book is by no means an elementary text. Though it presupposes no knowledge of conventional probability material, the reader should be well grounded in analysis. Kamke regards probability, not as a mysterious field closely allied to philosophy, but simply as a branch of function-theory directed toward infinite sequences of designated types. The book is compact; but it is extremely well written, and it is also surprisingly free from typographical imperfections. In the opinion of the reviewer, this is one of the most distinctive and important treatises on probability that have appeared in recent years.

E. L. Dodd

DUBISLAV ON FOUNDATIONS

Die Philosophie der Mathematik in der Gegenwart. By Walter Dubislav. (Philosophische Forschungsberichte, Heft 13.) Berlin, Junker und Dünnhaupt Verlag, 1932. vi+88 pp. R.M. 3.80.

Die Definition. By Walter Dubislav. Third, revised and augmented edition. (Beihefte der *Erkenntnis*, Heft 1.) Leipzig, Felix Meiner, 1931. viii+160 pp. R.M. 14.00.

In the little book *Die Philosophie der Mathematik in der Gegenwart*, Dubislav gives an incisive discussion of certain problems of the foundations of mathematics. The problems, "What is mathematics?" and "What can it claim to prove?" are subjects of vigorous controversy; our author gives the word to all contenders, and then has his own wise comments on the views of each.

After a presentation of elements of symbolic logic—a tool essential to further progress—he devotes a section to the "metamathematical" group of problems. The very fact that the word is used indicates Dubislav's place in the camp of Hilbert. As Ramsey puts it,* metamathematics consists of real, meaningful assertions about mathematics, which is itself meaningless; and, precisely, the main problems—those which this book deals with—are those of consistency, of the possibility of decision (general validity and fulfillability), and of completeness. The treatment in a work of this size must, of course, be fragmentary, and Dubislav chooses to emphasize the latest results obtained by the Hilbertians—for instance, Gödel's proof that all logico-mathematical calculi which comprise arithmetic contain indeterminable statements, one of which asserts the self-consistency of the system,† and Dubislav's own contribution to the problem of decision.

* *The Foundations of Mathematics*, p. 68.

† Kurt Gödel, *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme*, Monatshefte für Mathematik und Physik, vol. 38 (1931), p. 173.

The most valuable part of the final section is a clear statement of the views of six contending schools on the nature of mathematics. Dubislav's comment on these various doctrines is both penetrating and, in the reviewer's opinion, correct. The first theory, that most vigorously opposed, is the older, Kantian intuitionism (as it is named in the chapter on the subject-matter of mathematics); it is closely allied to criticism (Kritizismus) (the name used in the discussion of the foundations of the science). The Kantian doctrine holds that mathematics contains, over and above logic, synthetic judgments whose truth is guaranteed by their obviousness to every reasonable intellect. It must be said that the appeal to unanimity among sane minds, even if that unanimity could be ascertained, is an appeal to the principle of induction, itself an even less obvious principle than those which were so fundamental for Kant. "Criticism amounts to the assumption of Hilbertian metamathematics, supplemented by Kantian teachings on the nature of mathematics. For the adherents of criticism this adjunction is an essential deepening of Hilbert's conceptions; for the strict formalist an illegitimate adulteration of mathematics with partly mystical, partly metaphysical trains of thought."

If Kant held that man knows more things intuitively than most of us would now admit, Brouwer, also appealing to intuition, is so parsimonious as to the data which he trusts that admissible mathematics is tremendously complicated, while a large part of what we have called mathematics is denied all justification. That, of course, is no ground for rejecting it; more serious is the fact that Brouwer "is not in a position precisely to characterize those constructions—supposed to be independent of every language—of which mathematics is said to be built."

Other theories are: empiricism—"mathematics is that natural science which seeks to investigate the more general properties of objects"; conventionalism—"it is neither a natural science nor identifiable with logic, but a grand system of self-consistent demands, with their consequences"; logicism—"mathematics, as a branch of logic, is a system of tautological, unconditionally true statements."

All these theories are rejected—though with cordial recognition of their valuable contributions—in favor of formalism, for which mathematics is but a game with certain marks and rules. Logic is a part of this game, but a comparatively elementary one, for it lacks the infinity and selection axioms. The poverty of mathematics thus defined, its lack of contact with any reality, finds ample compensation in the wealth of real things which can replace the marks. "It is probable that formalism alone can succeed in bringing pure logic and pure mathematics to the state [freedom from 'real controversy'] described by Gauss. Hereto it is peculiarly adapted; for on its soil, free from metaphysics and mysticism, we can pursue logic and mathematics, with Fourier because of their great services to science, or with Jacobi pour l'honneur de l'esprit humain." Yet even formalism, it seems to us, can not dispense with an intuitional basis. That the substitution of b for x always changes $(a-x)$ into $(a-b)$ may seem far more certain than the principle of causality; but, if 99% of our acquaintances believed in a magical metamorphosis in connection with substitution, would our confidence remain?

The book concludes with two briefer chapters. One treats of the infinite, dwelling on the relations between Brouwer's and Hilbert's finitisms; the other tells how the various doctrines regard the relation of mathematics to the real world.

The second book to be considered, *Die Definition*, is quite as important a contribution to science—and is, naturally, broader in scope, as the definition of definition is not an affair of mathematics exclusively. The first, briefer, of the two sections of the book, reviews the main concepts of definition which have been proposed. There is the Aristotelian definition—determination of the essence of a thing (Wesensbestimmung). There is the Kantian definition—the determination of a concept (Begriffsbestimmung). A definition may, thirdly, be the report (Feststellung, nicht Festsetzung) of the meaning or use of a sign; this conception of the definition most closely approximates that of the dictionary. The fourth type of definition is the establishment (Festsetzung, nicht Feststellung) of the meaning or use of a sign. The first section closes with a brief history of its development. We find but one thing to criticize in this summary of definitions—that it is almost wholly devoid of those examples which so greatly clarify the use of highly abstract concepts.

The theory of definition most pleasing to Dubislav is an outgrowth of the Frege-Peano theory. Within a given discipline a definition is simply a rule for substitution. The method of such substitutions can, however, not be used without precautions—precautions, which, according to Dubislav, should be these: —that a definition be eliminable, and such that the repeated use of the substitution which it expresses in correct statements always leads to correct statements. A chief virtue of the Frege theory is the sharp distinction between the assertion and its clothing in language; its failings are the lack of clarity as to the permissible connections of signs in the definitions, and the confidence placed in "self-evident" axioms.

It is the formalistic "game theory" which gives a much more satisfactory account of the "Festsetzung" type of definition; it fills the gap in Frege's theory, giving an exact characterization of the external structure of a formula. In this section, the definition is regarded as the substitution of a mark for another collection of marks; since the main difficulty consists in the certain avoidance of paradoxes, there is an exposition of Russell's hierarchy.

We can not, however, neglect the "Feststellung" definition; for natural objects, as well as symbols, are defined. In this sense a definition coordinates sign with object. The dependence of "truth" on language, the preponderant importance of the negative result of an experiment, Hertz' analysis of a scientific theory (object→sign→sign→object), the approximate, statistical character of the verification of a law, are all needed as preliminaries to the discussion of the definition of an object. This second type of definition is described in detail, and illustrated by the definitions of order on a circle and of measurement on a line.

"Definition," as we saw at the outset, has yet other meanings. These, which have, in general, less to do with mathematics, are thoroughly treated by Dubislav, who ends his book with a clear summary of all types.

Dubislav is a master juggler of language. He can construct sentences whose analysis might grace an honors examination in grammar. The last three-quarters of a sentence on page 35 of the *Philosophie* reads: "Eine Definition durch Abstraktion besteht darin, dass man versucht, und zwar, ohne eine quasi-metaphysische Hilfsannahme zu machen, vergeblich versucht, aus einer Definitionsgleichung der Form $(fx=fy)=(xRy)Df$, wobei R eine transitive und symmetrische Relation ist, durch die ersichtlich das Zeichen $f\hat{z}$ nur im Zusammenhang mit anderen eingeführt wird, das betreffende Zeichen $f\hat{z}$ dennoch als ein vollständiges zu gewinnen, dem dann der durch die betreffende Definition angeblich in Gestalt eines idealen Objecktes schöpferisch erzeugte Gegenstand entsprechen soll."

In spite of the unusual powers revealed in this quotation, Dubislav rarely indulges in such telescopic delights. He writes with a firm, accurate style, revealing a mind of refreshing vigor.

In treatises on such abstract topics, one smiles, noting the warmth and enthusiasm with which the author defends his cold, unemotional concept of truth, and attacks the psychologists and metaphysicians who wish to be less restricted. "A purported statement about reality, based neither immediately nor indirectly on observations or their results, is a chimaera." "A science is no collection of sermons of salvation or messages of deliverance or rationalized myths or intellectual prose poems, based on emotional projections and their pseudo-rationalizations; it is, rather, a collection of statements which can, at least in principle, be tested by experience, together with the appropriate observations, experiments, and calculational transformations." Mathematical and philosophical thought have been definitely enriched by these contributions to their foundations.

E. S. ALLEN