CESÀRO SUMMABILITY OF DOUBLE SERIES

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1. *Definitions and Notation.* The familiar Cesàro transform of a double series $\sum_{i,j=1}^{\infty} u_{ij}$ is given by

$$\tilde{S}_{mn}^{(\alpha,\beta)} = S_{mn}^{(\alpha,\beta)} \Big/ \left[\binom{m+\alpha-1}{\alpha}\binom{n+\beta-1}{\beta}\right], \tag{1}$$

where

$$S_{mn}^{(\alpha,\beta)} = \sum_{i,j=1}^{m,n} \binom{m+\alpha-i}{\alpha}\binom{n+\beta-j}{\beta} u_{ij}. \tag{2}$$

The series $\sum u_{ij}$ is said to be summable $(C;\alpha,\beta)$ to S if we have $\lim_{m,n\to\infty}\tilde{S}_{mn}^{(\alpha,\beta)} = S$; to be bounded $(C;\alpha,\beta)$ if $|\tilde{S}_{mn}^{(\alpha,\beta)}| <$ const. for all values of m and n; and to be ultimately bounded $(C;\alpha,\beta)$ if $\limsup_{m,n\to\infty}|\tilde{S}_{mn}^{(\alpha,\beta)}| < \infty$. This definition holds for all values of α and β, real or complex, except negative integers, the binomial coefficients being defined as usual in terms of the gamma function; however we shall be concerned only with real orders greater than -1.

A special but important type of double series is that for which u_{ij} is factorable, say

$$u_{ij} = v_i w_j, \qquad (i, j = 1, 2, 3, \cdots). \tag{3}$$

Defining

$$V_m^{(\alpha)} = \sum_{i=1}^{m}\binom{m+\alpha-i}{\alpha} v_i;\quad W_n^{(\beta)} = \sum_{j=1}^{n}\binom{n+\beta-j}{\beta} w_j, \tag{4}$$

we have, by (2), when (3) holds, $S_{mn}^{(\alpha,\beta)} = V_m^{(\alpha)} W_n^{(\beta)}$, so that, by (1),

$$\tilde{S}_{mn}^{(\alpha,\beta)} = \tilde{V}_m^{(\alpha)} \tilde{W}_n^{(\beta)}, \tag{5}$$

where the factors in the right member of (5) are, respectively, the (C,α) transform of $\sum v_i$ and the (C,β) transform of $\sum w_j$.

2. *Examples.* The relation (5) enables us to obtain very easily examples illustrating the following statements.

THEOREM 1. *There is a series* $\sum u_{ij}$ *which is summable and bounded* $(C;\ \alpha,\beta)$ *for every* $\alpha>0$, $\beta>0$, *while* (a) *each row and column of* $\sum u_{ij}$ *has bounded partial sums and* (b) *each row and column of* $\sum u_{ij}$ *is non-summable* $(C,\ \gamma)$ *for every* $\gamma>-1$.

Let $v_1=-1$ and $v_i=2(-1)^i$ when $i>1$; then $\sum_{i=1}^m v_i=(-1)^m$ and $\sum v_i$ is, as is well known, summable $(C,\ \delta)$ to 0 for every $\delta>0$.* Let $\sum w_j$ be any series whose partial sums are bounded, and which is non-summable $(C,\ \gamma)$ for every $\gamma>-1$.† It is easy to show that the series whose general term is given by

$$u_{ij} = v_i w_j + w_i v_j$$

is summable $(C;\alpha,\beta)$ to 0, and satisfies the other conditions of Theorem 1.

THEOREM 2. *Corresponding to each pair of numbers* α *and* β, $\alpha>-1$, $\beta>-1$, *there is a series* $\sum u_{ij}$ *which is summable and bounded* $(C;\ \alpha,\beta)$ *while* (a) *each row* [*column*] *is unbounded* $(C,\beta-\delta)$ $[(C,\alpha-\delta)]$, (b) *each row* [*column*] *is non-summable but bounded* $(C,\ \beta)$ $[(C,\ \alpha)]$ *and* (c) *each row* [*column*] *is summable* $(C,\beta+\delta)$ $[(C,\alpha+\delta)]$, *for every* $\delta>0$.

Let $v_1=-1$ and $v_i=2(-1)^i$ when $i>1$, as before. Corresponding to a number $\gamma>-1$, let $\sum w_i^{(\gamma)}$ be the series having

$$\sum_{i=1}^{p}(-1)^i\binom{i+\gamma-1}{\gamma}, \quad (p = 1, 2, 3, \cdots),$$

for its sequence of partial sums. Then $\sum w_i^{(\gamma)}$ has‡ an unbounded $(C,\ \gamma-\delta)$ transform, a non-convergent but bounded $(C,\ \gamma)$ transform, and a convergent $(C,\ \gamma+\delta)$ transform. These facts and properties of $\sum v_i$ enable us to show that the series whose general term is given by $u_{ij}=v_i w_j^{(\beta)}+w_i^{(\alpha)}v_j$ is summable $(C;\alpha,\beta)$ to 0 and fulfills the other conditions of Theorem 2.

* Σv_i is bounded $(C,\ 0)$ and summable $(C,\ 1)$ to 0; it is therefore summable $(C,\ \delta)$ to 0 for every $\delta>0$. See Zygmund, *Sur un théorème de la théorie de la sommabilité*, Mathematische Zeitschrift, vol. 25 (1926), p. 291.

† An example of such a series is Σw_j, where w_j is the coefficient of x^j in the series $\sum_{\nu=0}^{\infty}(-1)^\nu x^{\nu!}$. See Hardy, *On certain oscillating series*, Quarterly Journal of Mathematics, vol. 38 (1906–7), p. 286. Hardy's result shows that $\sum w_j$ is not Abel summable, and non-summability $(C,\ \gamma)$ for every $\gamma>-1$ follows.

‡ See Knopp, *Unendliche Reihen*, 3d edition, 1931, p. 496, Ex. 2. The discussion is given for $\gamma=k$, an integer, but it holds also for any real $\gamma>-1$.

3. *Necessary Conditions for Summability.* We now give two necessary conditions for summability of double series.

THEOREM 3. *If a double series is ultimately bounded $(C; k, l)$, k and l being fixed positive integers, then each sufficiently advanced row [column] is bounded (C, l) $[(C, k)]$.*

By the ultimate boundedness $(C; k, l)$ there exist constants K, M, and N such that $|\tilde{S}_{mn}^{(k,l)}| < K$ for $m > M$, $n > N$. Fix $m > M + k + 1$; then for every $n > N$, $|\tilde{S}_{m-r,n}^{(k,l)}| < K$, for $r = 0, 1, 2, \cdots, k+1$. Moreover, we have

$$|S_{m-r,n}^{(k,l)}| \Big/ \binom{m+k-1}{k}\binom{n+l-1}{l}$$

$$= |\tilde{S}_{m-r,n}^{(k,l)}| \binom{m+k-r-1}{k} \Big/ \binom{m+n-1}{k} < K,$$

and

$$\left| \sum_{r=0}^{k+1} (-1)^r \binom{k+1}{r} S_{m-r,n}^{(k,l)} \Big/ \binom{m+k-1}{k}\binom{n+l-1}{l} \right|$$

$$< K \sum_{r=0}^{k+1} \binom{k+1}{r} = 2^{k+1} K.$$

The product of $\binom{m+k-1}{k}$ by the sum on the left is the (C, l) transform of the mth row of the double series.* This row is a simple series which is ultimately bounded (C, l) and therefore is bounded (C, l) as was to be shown.

THEOREM 4. *If a double series is summable $(C; k, l)$, k and l being fixed positive integers, then each sufficiently advanced row [column] is either* (a) *summable $(C, l+\delta)$ $[(C, k+\delta)]$ for every $\delta > 0$ or* (b) *non-summable (C, γ) for every $\gamma > -1$.*

This follows from Theorem 3 and the result of Zygmund, loc. cit.

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* This is seen by equating coefficients of $x^m y^n$ in the identity

$$(1-x)^{k+1} \sum_{i,j=1}^{\infty} S_{ij}^{(k,l)} x^i y^j = (1-y)^{-(l+1)} \sum_{i,j=1}^{\infty} u_{ij} x^i y^j.$$