ON CERTAIN CHARACTERISTICS OF k-DIMENSIONAL VARIETIES IN r-SPACE

BY B. C. WONG

An algebraic variety of k dimensions in r-space has numerous characteristics besides its order n. The characteristics of algebraic curves and surfaces and the relations they satisfy are known. In this paper we consider a variety V_k of dimension k greater than 2. Assuming it to be the complete intersection of $r-k$ hypersurfaces of orders $n_1, n_2, \cdots, n_{r-k}$ respectively in S_r, we derive the formulas for a few of its characteristics in terms of the n's and incidentally obtain the relations connecting them. To avoid unnecessary length of discussion we consider somewhat in detail the V_3 in S_7 only and then give the results without demonstration for V_k in S_r.* The method here employed is the familiar one of complete degeneration which we have repeatedly made use of elsewhere in dealing with problems of similar nature.†

Now for the purpose of enumerating the characteristics of V_k and obtaining their relations we may regard the variety as belonging to an S_{2k+1}, for a V_k belonging to an S_r where $r>2k+1$ possesses no characteristics not possessed by a V_k of S_{2k+1}. If we project V_k from a general S_{t-1} of S_{2k+1} on to an S_{2k+1-t} $[0 \leqq t \leqq k]$ of S_{2k+1}, we have for projection a $V_k^{(t)}$ possessing a double $(t-1)$-dimensional variety $V_{t-1}^{b_{t-1}}$ of order b_{t-1} and a pinch $(t-2)$-dimensional variety $V_{t-2}^{j_{t-2}}$ of order j_{t-2} lying on $V_{t-1}^{b_{t-1}}$. From a general point of S_{2k+1-t} we can construct ∞^t lines forming a $(t+1)$-dimensional cone of order b_t each meeting $V_k^{(t)}$ in two distinct points. We say that $V_k^{(t)}$ has an apparent double $V_t^{b_t}$ of order b_t. Again, from a general point of S_{2k+1-t} a t-dimensional cone of ∞^{t-1} lines of order j_{t-1} can be constructed tangent to

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* Some work has been done along this line. See C. Segre, *Mehrdimensionale Räume*, Encyklopädie der Mathematischen Wissenschaften, III$_2$, 7, pp. 922–927.

† B. C. Wong, *On the number of apparent multiple points of varieties in hyperspace*, this Bulletin, vol. 36, pp. 102–106; and *On surfaces in spaces of four and five dimensions*, this Bulletin, vol. 36, pp. 861–866.

$V_k^{(t)}$. The projection $V_k^{(t+1)}$ of $V_k^{(t)}$ in an S_{2k-t} has a double variety $V_t^{b_t}$ which is the projection of the apparent double variety $V_t^{b_t}$ of $V_k^{(t)}$ and a pinch variety $V_{t-1}^{j_{t-1}}$ which is the section of the tangent t-dimensional cone of $V_k^{(t)}$ by S_{2k-t}.

If we allow t to take on all the values from 0 to k, we obtain $2k$ characteristics $b_0, b_1, \cdots, b_{k-1}; j_0, j_1, \cdots, j_{k-1}$. For $t=0$, we have V_k itself, and it has b_0 apparent double points. For $t=1$, the projection V_k' in S_{2k} has b_0 improper double points and an apparent double curve of order b_1. From a general point of S_{2k} we can draw j_0 tangent lines to V_k'. Now if $t=2$, the projection V_k'' in S_{2k-1} has a double curve of order b_1 on which lie j_0 pinch points and an apparent double surface of order b_2, and is such that the cone of the ∞^1 tangent lines drawn from a general point of S_{2k-1} is of order j_1. And so on for the other values of t. For $t=k$, the projection $V_k^{(k)}$ in S_{k+1} has a double variety $V_{k-1}^{b_{k-1}}$ and a pinch variety $V_{k-2}^{j_{k-2}}$ lying on $V_{k-1}^{b_{k-1}}$. The characteristic j_{k-1} is the order of the tangent k-dimensional cone and it is also the class of the curve in which a plane of S_{k+1} meets $V_k^{(k)}$.

There are numerous other characteristics of the variety V_k such as the orders of its various manifolds of multplicities higher than 2 and the ranks of its different sections by subspaces of S_{2k+1}. With these we are not at present concerned and we are here concerned only with the b's and the j's just enumerated. Now we take the case $k=3$, that is, the V_3 in S_7.

By the method of complete degeneration we regard the four hypersurfaces of orders n_1, n_2, n_3, n_4 in S_7 which intersect in the V_3 we are studying as being composed entirely of hyperplanes: $A_1, A_2, \cdots, A_{n_1}$; $B_1, B_2, \cdots, B_{n_2}$; $C_1, \cdots, C_{n_3}$; $D_1, \cdots, D_{n_4}$. The V_3 of intersection is then composed of $n=n_1n_2n_3n_4$ 3-spaces $(A_{i_1}B_{i_2}C_{i_3}D_{i_4})$ $[i_j=1, 2, \cdots, n_j]$. We write $(x_1x_2x_3x_4)$ in place of $(A_{i_1}B_{i_2}C_{i_3}D_{i_4})$. This symbol for any set of particular values of the x's represents a particular S_3 belonging to the decomposed V_3. Thus, the set (1432) represents the S_3 common to A_1, B_4, C_3, D_2. The totality of all the $n=n_1n_2n_3n_4$ sets $(x_1x_2x_3x_4)$ for all the integral values of x_i from 1 to n_i $[i=1, 2, 3, 4]$ will be taken to be the symbolic representation of V_3.

To deal with the different apparent double varieties of V_3 is to deal with the different kinds of pairs of S_3's of the decomposed V_3, that is, with the different kinds of pairs of sets of values

$(x_1x_2x_3x_4)$. There are in all $n(n-1)/2$ pairs and they fall under four types. Under type I we have all those pairs each of which is such that the elements or x's of one set are all different from the corresponding elements or x's of the other, as for example (1111), (2324). Any such pair represents a pair of non-incident S_3's belonging to the decomposed V_3 and the number of such pairs is, as can easily be verified,

$$2^3\binom{n_1}{2}\binom{n_2}{2}\binom{n_3}{2}\binom{n_4}{2}.$$

Now type II consists of all those pairs in each of which three elements of one set are all different from the three corresponding elements of the other. Examples are (1111), (1223); (1234), (2414). A pair of this kind represents two S_3's in the decomposed V_3 having a point in common. The number of pairs belonging to this type is given by the symmetric function

$$2^2\sum n_1\binom{n_2}{2}\binom{n_3}{2}\binom{n_4}{2}.$$

For type III we have all those pairs each consisting of sets two of whose corresponding elements are different and two alike, as (1111), (1123). There are

$$2\sum n_1n_2\binom{n_3}{2}\binom{n_4}{2}$$

such pairs each representing two S_3's in the decomposed V_3 with a line in common.

The remaining

$$\sum n_1n_2n_3\binom{n_4}{2}$$

pairs of sets, that is, all those pairs each having three elements in one set and the three corresponding elements in the other alike, are said to belong to type IV. A pair of this type represents a pair of S_3's having a plane in common and belonging to the decomposed V_3.

Now from a given point in S_7 only one line can be drawn incident with two non-incident S_3's. We say that these two S_3's have a point of apparent intersection. The number of pairs of non-incident S_3's in the decomposed V_3 is the number of apparent double points of V_3 before decomposition. Being equal to the number of pairs of sets of values of the x's belonging to type I, it is given by

$$b_0 = 2^3\binom{n_1}{2}\binom{n_2}{2}\binom{n_3}{2}\binom{n_4}{2}.$$

Projecting the decomposed V_3 on to an S_5, we find that all those pairs of S_3's both non-incident and incident in a point project into pairs of S_3's each having an actual line of incidence. All the lines so obtained form the degenerate double curve in the projection of the decomposed V_3. Hence, the order of the double curve on the projection V_3'' of V_3 before degeneration is, from the results obtained for type I and type II,

$$b_1 = 2^3\binom{n_1}{2}\binom{n_2}{2}\binom{n_3}{2}\binom{n_4}{2} + 2^2\sum n_1\binom{n_1}{2}\binom{n_2}{2}\binom{n_3}{2}.$$

A pinch point on V_3'' is given by a tangent plane of V_3 passing through a given line l in S_7. If V_3 is completely degenerated, the given line l and the point of incidence of the two S_3's of a pair represented by a pair of sets of type II determine a plane which is to be regarded as two tangent planes of V_3 passing through l. Therefore, we have

$$j_0 = 2^3\sum n_1\binom{n_2}{2}\binom{n_3}{2}\binom{n_4}{2}$$

for the number of pinch points on the projection V_3'' in S_5.

Now a given plane of S_7 and the line common to two S_3's of a pair whose representation belongs to type III determine ∞^1 3-spaces each of which is to be considered as two tangent 3-spaces of V_3. If V_3 is projected on to an S_4, every such tangent 3-space meets S_4 in a pinch point of V_3'''. The order of the locus of pinch points on V_3''' is twice the number of pairs of sets of the x's belonging to type III and hence we have

$$j_1 = 2^2\sum n_1 n_2\binom{n_3}{2}\binom{n_4}{2}.$$

In the projection in S_4 of the decomposed V_3 we find that the number of planes each of which is the projection of the plane of apparent intersection of the S_3's of a pair whose representation belongs to either type I or II or III is the order of the double surface on V_3''' and is given by

$$b_2 = 2^3\binom{n_1}{2}\binom{n_2}{2}\binom{n_3}{2}\binom{n_4}{2} + 2^2\sum n_1\binom{n_2}{2}\binom{n_3}{2}\binom{n_4}{2}$$
$$+ 2\sum n_1 n_2 \binom{n_3}{2}\binom{n_4}{2}.$$

It is not difficult to see that the order j_2 of the tangent 3-dimensional cone of V_3''' is twice the number of pairs in type IV and we have

$$j_2 = 2\sum n_1 n_2 n_3 \binom{n_4}{2}.$$

Incidentally we have the obvious relations

$$2b_1 = 2b_0 + j_0, \quad 2b_2 = 2b_1 + j_1 = 2b_0 + j_0 + j_1$$

and

$$n(n-1) = 2b_2 + j_2 = 2b_0 + j_1 + j_2.$$

It is to be noticed that if one of the four n's is unity, then $b_0 = 0$, that is, if V_3 is the complete intersection of three general V_5's in an S_6, it cannot have improper double points. If two of the four n's are equal to unity, then we have, in addition to b_0 vanishing, $b_1 = 0$ and $j_0 = 0$. This means that a V_3 of complete intersection of two general V_4's in an S_5 has no double curve nor pinch points.

Now we write the results for the b's and the j's of V_k in S_r for $r \geqq 2k+1$. If V_k is the complete intersection of $r-k$ general hypersurfaces of orders $n_1, n_2, \cdots, n_{r-k}$ respectively, we find by arguments exactly similar to those for the case $k=3$ the following formulas:

$$b_{k-q} = \sum_{i=1}^{j-k-q} 2^{q-1+i} \sum n_1 n_2 \cdots n_{r-k-q-i}\binom{n_{r-k-q-i+1}}{2}$$
$$\cdot \binom{n_{r-k-q-i+2}}{2} \cdots \binom{n_{r-k}}{2},$$

and

$$j_{k-q} = 2^q \sum n_1 n_2 \cdots n_{r-k-q}\binom{n_{r-k-q+1}}{2}\binom{n_{r-k-q+2}}{2}\cdots\binom{n_{r-k}}{2}.$$

For $q = 1$, we have

$$b_{k-1} = \sum_{i=1}^{r-k-1} 2^i \sum n_1 n_2 \cdots n_{r-k-1-i} \binom{n_{r-k-i}}{2} \binom{n_{r-k-i+1}}{2} \cdots \binom{n_{r-k}}{2}$$
$$= \tfrac{1}{2} n_1 n_2 \cdots n_{r-k} (n_1 n_2 \cdots n_{r-k} - \sum n_1 + r - k - 1),$$

which is the number of apparent double points on the curve in an S_3 into which the curve of intersection of V_k by an S_{r-k+1} is projected, and

$$j_{k-1} = 2 \sum n_1 n_2 \cdots n_{r-k-1} \binom{n_{r-k}}{2} = n_n \cdots n_{r-k} \sum (n_1 - 1),$$

which is the rank of the curve of intersection of V_k by an S_{r-k+1}.

By assigning to q all the different values from 1 to k, we obtain all the formulas for the $2k$ characteristics $b_0, b_1, \cdots, b_{k-1}$; $j_0, j_1, \cdots, j_{k-1}$. We have also the relations

$$\begin{aligned} 2b_{k-q} &= j_{k-q-1} + 2b_{k-q-1} \\ &= j_{k-q-1} + j_{k-q-2} + 2b_{k-q-2} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &= j_{k-q-1} + j_{k-q-2} + \cdots + j_1 + j_0 + 2b_0, \end{aligned}$$

and

$$n(n-1) = j_{k-1} + j_{k-2} + \cdots + j_{k-q} + 2b_{k-q}.$$

We wish to add that, if $r < 2k+1$ and if $2k+1-r=d$, we have $b_0 = b_1 = \cdots = b_{d-1} = 0$ and also $j_0 = j_1 = \cdots = j_{d-2} = 0$.

THE UNIVERSITY OF CALIFORNIA