# ON UNIT-ZERO BOOLEAN REPRESENTATIONS OF OPERATIONS AND RELATIONS*

BY B. A. BERNSTEIN

1. *Introduction.* Consider an algebra $(K, +, \times)$, such as ordinary real algebra, in which there are two elements "0" and "1" having the properties that, for any element $a$,

$$(1) \qquad a + 0 = 0 + a = a, \; a1 = 1a = a.$$

Let

$$(x_1, x_2, \cdots, x_m; a_1, a_2, \cdots, a_m)$$

denote a *unit-zero function with respect to the sequence of m elements*, $a_1, \cdots, a_m$ of $K$, that is, a function $f(x_1, x_2, \cdots, x_m)$ of $m$ elements $x_1, x_2, \cdots, x_m$ such that $f=1$ or 0, according as the equalities, $x_i = a_i$, $(i=1, 2, \cdots, m)$, all hold or do not all hold. Accordingly, $(x; a)$ will denote a *unit-zero function with respect to a*, that is, a function $f(x)$ such that $f(x)=1$ or 0, according as $x=a$ or $x \neq a$. Then the following propositions (2)–(4) evidently hold:

$$(2) \; (x_1, x_2, \cdots, x_m; a_1, a_2, \cdots, a_m) = (x_1; a_1)(x_2; a_2) \cdots (x_m; a_m);$$

$$(3) \qquad a(x_1, x_2, \cdots, x_m; a_1, a_2, \cdots, a_m) = a \text{ or } 0,$$

according as $x_i = a_i$, $(i = 1, 2, \cdots, m)$, all hold or do not all hold;

$$(4) \; a(x_1, x_2, \cdots, x_m; a_1, a_2, \cdots, a_m) + b(x_1, x_2, \cdots, x_m; b_1, b_2, \cdots, b_m) = a, \text{ or } b, \text{ or } 0,$$

according as $x_i = a_i$ all hold, or $x_i = b_i$ all hold, or neither $x_i = a_i$ all hold nor $x_i = b_i$ all hold, $(i = 1, 2, \cdots, m; a_i \neq b_i$ for some $i)$. In a previous paper† propositions (1)–(4) were made the basis of a method of obtaining arithmetic representations of arbitrary operations and relations in a finite class of elements. Since

---

* Presented to the Society, April 11, 1931.

† B. A. Bernstein and N. Debely, *A practical method for the modular representation of finite operations and relations*, this Bulletin, vol. 38 (1932), pp. 110–114.

propositions (1)–(4) also hold when the symbols belong to Boolean algebra, the question naturally arises: To what extent can unit-zero functions be used analogously to obtain *Boolean* representations of arbitrary operations and relations? The object of the present paper is to answer this question.

2. *Determination of Boolean Unit-Zero Algebras.* The possibility of representing arbitrary operations and relations by unit-zero functions of an algebra hinges on the existence in this algebra of a unit-zero function for every sequence of $m$ of its elements. Let us call an algebra which has a unit-zero function for every sequence of $m$ of its elements a *unit-zero algebra.* I proceed first to determine all Boolean unit-zero algebras.

This determination is made easy by noting at the outset that a unit-zero Boolean function must satisfy proposition (2) above and also that it must be single-valued. We therefore need to look only for Boolean unit-zero functions $f(x)$ of a *single* variable $x$ of the form*

$$(x;\, a) = (1;\, a)x + (0;\, a)x'. \tag{5}$$

From (5) we see, by putting $a=0,\, 1$, that in a Boolean algebra of *two* elements, $x$ is the unit-zero function of $x$ with respect to 1, and $x'$ is the unit-zero function of $x$ with respect to 0; in symbols,

$$(x;\, 1) = x, \quad (x;\, 0) = x'. \tag{6}$$

We have, then, that *a two-element Boolean algebra is a unit-zero algebra, the unit-zero functions of one variable $x$ being given by* (6).

By (2) and (6), *all* the unit-zero functions of a two-element Boolean algebra can be readily written down. Thus, the unit-zero functions of two variables $x$, $y$ are given by

$$\begin{aligned} (x,\, y;\, 1,\, 1) &= xy, \quad (x,\, y;\, 1,\, 0) = xy', \\ (x,\, y;\, 0,\, 1) &= x'y, \quad (x,\, y;\, 0,\, 0) = x'y'. \end{aligned} \tag{7}$$

In general, *the unit-zero functions of $m$ variables are the $2^m$ constituents in the normal development of* 1 *with respect to the $m$ variables.*

---

* The usual Boolean notations are employed: $a+b$, $ab$, $a'$, 0, 1 are respectively the *sum* of $a$ and $b$, the *product* of $a$ and $b$, the *negative* of $a$, the *zero* element, the *whole.*

Let us now consider a Boolean algebra $A$ of *more* than two elements. $A$ must have an element $e \neq 0, 1$. Suppose, first, that $A$ has a unit-zero function $f(x)$, of form (5), with respect to $e$. Then

$$\text{(i)} \qquad f(e) = 1, f(0) = 0, f(1) = 0, \qquad (e \neq 0, 1).$$

But (i) is inconsistent with (5). Hence, *our algebra $A$ has no unit-zero function with respect to a sequence containing the element $e$.*

Suppose, next, that the algebra $A$ has a unit-zero function $f(x)$, of form (5) with respect to 0. Then

$$\text{(ii)} \qquad f(0) = 1, f(1) = 0, f(e) = 0, \qquad (e \neq 0, 1).$$

Hence, by (5),

$$\text{(iii)} \qquad f(x) = x', f(e) = 0, \qquad (e \neq 0, 1).$$

But equations (iii) are inconsistent. Hence, *our algebra $A$ has no unit-zero function with respect to a sequence containing the element* 0.

Similarly, *our algebra $A$ has no unit-zero function with respect to a sequence containing the element* 1. Hence, *a Boolean algebra of more than two elements has no unit-zero functions at all.*

Our main result is, then, the following theorem.

THEOREM A. *The only Boolean unit-zero algebra is a two-element Boolean algebra.*

3. *Dual Considerations.* By the Principle of Duality in Boolean algebras each of the foregoing propositions about unit-zero Boolean functions has a dual proposition corresponding to it. To state these duals, let me use the notion of *zero-unit function* (to be distinguished from *unit-zero function*). By a *zero-unit function of $x_1, x_2, \cdots, x_m$ with respect to the sequence* $a_1, a_2, \cdots, a_m$, symbolized by

$$[x_1, x_2; \cdots, x_m; a_1, a_2, \cdots, a_m],$$

let us mean a function $f(x_1, x_2, \cdots, x_m)$ such that $f=0$ or 1, according as $x_i = a_i$, $(i = 1, 2, \cdots, m)$, all hold or do not all hold. The duals of (2), (3), and (4) are, then, respectively (2′), (3′), and (4′) following:

(2′) $[x_1, x_2, \cdots, x_m; a_1, a_2, \cdots, a_m]$
$$= [x_1; a_1] + [x_2; a_2] + \cdots + [x_m; a_m];$$

(3′) $a + [x_1, x_2, \cdots, x_m; a_1, a_2, \cdots, a_m] = a$ or 1,

according as $x_i = a_i$, $(i = 1, 2, \cdots, m)$, all hold or do not all hold;

(4′) $\{a + [x_1, x_2, \cdots, x_m; a_1, a_2, \cdots, a_m]\}$
$$\cdot \{b + [x_1, x_2, \cdots, x_m; b_1, b_2, \cdots, b_m]\} = a, \text{ or } b, \text{ or } 1,$$

according as $x_i = a_i$ all hold, or $x_i = b_i$ all hold, or neither $x_i = a_i$ all hold nor $x_i = b_i$ all hold, $(i = 1, 2, \cdots, m; a_i \neq b_i$ for some $i)$.

The dual of Theorem A is

THEOREM A′. *The only zero-unit Boolean algebra is a two-element Boolean algebra.*

For a *two-element* Boolean algebra we have, further:

(6′) $[x; 0] = x, \quad [x; 1] = x';$

(7′) $[x, y; 0, 0] = x + y, \quad [x, y; 0, 1] = x + y',$
$$[x, y; 1, 0] = x' + y, \quad [x, y; 1, 1] = x' + y'.$$

In general, *the zero-unit functions of m variables are the* $2^m$ *factor-constituents in the dual normal development of* 0 *with respect to the m variables.*

Propositions (2′)–(7′) will be used below in the representation of operations that do not satisfy the condition of closure.

4. *Representations.* It is now clear to what extent we can apply unit-zero Boolean functions in the representation of arbitrary operations and relations. From Theorem A, we have

THEOREM B. *A unit-zero Boolean representation of arbitrary operations and relations is possible when and only when the class consists of two elements.*

For a two-element class $K$, the theory of Boolean representation follows from propositions (2)–(7) and their duals. If we denote the two $K$-elements by the Boolean symbols 0, 1, the representations of all operations $O$ and relations $R$ in $K$ are covered by the cases 1–3 following.

CASE 1. *$O$ an $m$-ary operation satisfying the condition of closure.* There is a $K$-element, 0 or 1, for every sequence $e_1, e_2, \cdots, e_m$ taken from $K$. Let the sequences to which 1 corresponds be

(i) $\alpha_{11}, \alpha_{12}, \cdots, \alpha_{1m}; \alpha_{21}, \alpha_{22}, \cdots; \alpha_{2m}; \cdots; \alpha_{k1}, \alpha_{k2}, \cdots, \alpha_{km}$.

The representation of $O$ is the Boolean function

$$\sum_{i=1}^{k}(x_1, x_2, \cdots, x_m; \alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{im}). \tag{8}$$

CASE 2. *$O$ an $m$-ary operation not satisfying the closure condition.* There are sequences in $K$ to which no $K$-elements correspond. Let these sequences be

(ii) $\beta_{11}, \beta_{12}, \cdots, \beta_{1m}; \beta_{21}, \beta_{22}, \cdots, \beta_{2m}; \cdots; \beta_{k1}, \beta_{k2}, \cdots, \beta_{km}$.

Consider the operation $O'$ obtained from $O$ by assigning a $K$-element, 0 for convenience, to each of the sequences (ii). Let $\phi(x_1, x_2, \cdots, x_m)$, obtained as in Case 1, be the representation of $O'$. Then the representation of $O$ is the function

$$\phi(x_1, x_2, \cdots, x_m) + \sum_{i=1}^{k} 0/[x_1, x_2, \cdots, x_m; \beta_{i1}, \beta_{i2}, \cdots, \beta_{im}], \tag{9}$$

where $a/b$ means the unique $K$-element $q$ satisfying the condition $bq = a$.*

CASE 3. *$R$ an $m$-adic relation.* Let the sequences which do *not* satisfy $R$ be

(iii) $\gamma_{11}, \gamma_{12}, \cdots, \gamma_{1m}; \gamma_{21}, \gamma_{22}, \cdots, \gamma_{2m}; \cdots; \gamma_{k1}, \gamma_{k2}, \cdots, \gamma_{km}$.

Then the representation of $R$ is the Boolean equation

$$\sum_{i=1}^{k}(x_1, x_2, \cdots, x_m; \gamma_{i1}, \gamma_{i2}, \cdots, \gamma_{im}) = 0.\dagger \tag{10}$$

Of course, by the Duality Principle, the theory of representation can be stated primarily in terms of zero-unit functions instead of unit-zero functions.

5. *Illustrations.* The following illustrations, one for each of the above three cases, will make the theory of representation quite clear.

$\alpha$. Let $O$ be the operation defined by

---

* For a two-element Boolean algebra the *quotient* can be defined precisely as in ordinary algebra.

† Instead of 0, we can use 1 in (10), provided (i) are the sequences which *do* satisfy $R$.

(i)

| | 0 | 1 |
|---|---|---|
| 0 | 1 | 0 |
| 1 | 0 | 1 |

Its representation is

(ii) $(x, y; 0, 0) + (x, y; 1, 1) \equiv x'y' + xy.$

$\beta$. Let $O$ be the operation

(iii)

| | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | – | – |

where the blanks indicate that there are no $K$-elements corresponding to the sequences 1, 0; 1, 1.

Consider the operation $O'$ defined by

(iv)

| | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 0 | 0 |

By Case 1, the representation of $O'$ is

(v) $x'y.$

Hence, the representation of $O$ is

(vi) $x'y + 0/[x, y; 1, 0] + 0/[x, y; 1, 1]$
$$\equiv x'y + 0/(x' + y) + 0/(x' + y').$$

$\gamma$. Let $R$ be a relation defined by

(vii)

| | 0 | 1 |
|---|---|---|
| 0 | − | + |
| 1 | + | − |

where "+" indicates that $R$ holds and "−" indicates that $R$ does not hold. Its representation is the equation

(viii) $(x, y; 0, 0) + (x, y; 1, 1) \equiv x'y' + xy = 0.$*

THE UNIVERSITY OF CALIFORNIA

* For a complete set of Boolean representations of binary operations and dyadic relations in a two-element class, obtained from considerations other than the above, see my *Complete sets of representations of two-element algebras*, this Bulletin, vol. 30 (1924), pp. 24–30.