AN INVERSIVE ALGORITHM*

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Viggo Brun‡ has given an algorithm for calculating directly the nth prime number from certain values of the function $\pi(x)$, the number of primes $\leqq x$. It is the purpose of this note to show that this algorithm is not peculiar to primes, but on the contrary, may be used to exhibit the nth member of any infinite class C of positive integers. With this degree of generality it is possible to use the algorithm to obtain identities between numerical functions.

The algorithm may be described as follows. Associated with the class C is the enumerative function $\theta(x)$ giving the number of members of C which are $\leqq x$. If n is any positive integer we form the sequence

$$n_0, n_1, n_2, \cdots, n_r, \cdots, \tag{1}$$

whose terms are defined as follows:

$$\begin{aligned} n_0 &= n, \\ n_1 &= n - \theta(n_0), \\ n_2 &= n - \theta(n_0 + n_1), \\ n_3 &= n - \theta(n_0 + n_1 + n_2), \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

$$n_r = n - \theta(s_r), \tag{2}$$

where, for brevity, we have written

$$s_r = n_0 + n_1 + n_2 + \cdots + n_{r-1}.$$

We have then the following theorem.

THEOREM. *The terms of the sequence* (1) *do not increase, and ultimately become and remain zero. If k is the rank of the first zero term of* (1), *then s_k is the nth member of the class C.*

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‡ Kongelige Norske Videnskaps-Selskabet, vol. 4, pp. 66–69.

PROOF. We first show that the terms of (1) are non-negative. This is true of $n_0 = n$. If it is true of n_{r-1} we may show it true for n_r as follows. Consider the difference

$$(3) \quad n_{r-1} - n_r = \theta(s_r) - \theta(s_{r-1}) = \theta(s_{r-1} + n_{r-1}) - \theta(s_{r-1}).$$

If $n_{r-1} = 0$, it follows that $n_r = 0$. If $n_{r-1} > 0$, (3) gives the number of members of C lying among the n_{r-1} consecutive integers

$$s_{r-1} + 1,\ s_{r-1} + 2,\ \cdots,\ s_r.$$

Hence we have $0 \leqq n_{r-1} - n_r \leqq n_{r-1}$. The first inequality shows that the terms of (1) do not increase. This second inequality implies $n_r \geqq 0$. We have just seen that if a term of (1) is zero all further terms vanish. Hence to prove the first part of the theorem it is sufficient to show that the assumption that all terms of (1) are positive leads to a contradiction. This assumption implies that s_r increases indefinitely with r. But C is an infinite class. Hence for r sufficiently large, $n_r = n - \theta(s_r) < 0$ contrary to fact. To complete the proof set $r = k$ and $n_r = 0$ in (2) and (3). Then (3) becomes $n_{k-1} = \theta(s_{k-1} + n_{k-1}) - \theta(s_{k-1})$. This tells us that all the numbers

$$s_{k-1} + 1,\ s_{k-1} + 2,\ \cdots,\ s_k$$

belong to C. In particular s_k belongs to C. But by (2), $\theta(s_k) = n$. Hence s_k is the nth member of C.

From the equations defining the terms of (1) one may eliminate the first terms and obtain in this way an expression for n_r which involves only n and θ. In general this expression is very complicated. In particular cases it may be simplified and we obtain from the algorithm a certain identity. One example will suffice to illustrate the method.

Let the set C consist of all positive even integers. Then $\theta(x) = [x/2]$. In this case it is not difficult to show that

$$n_r = \left[\frac{n + 2^{r-1}}{2^r}\right].$$

Hence we have at once the identity

$$n = \left[\frac{n+1}{2}\right] + \left[\frac{n+2}{4}\right] + \left[\frac{n+4}{8}\right] + \cdots.$$

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