ANALYTIC STUDY OF RATIONAL QUINTIC SURFACES HAVING NO MULTIPLE CURVES

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1. *Introduction.* The purpose of this paper is to derive the equations of certain of the rational quintic surfaces without multiple curves discussed synthetically by Montesano.* The equations of the surfaces are found by applying Cremona transformations to certain well known rational surfaces of order three or four.

2. *Surface of Order Five with Four Triple Points.* This surface is the transform by the cubic transformation T_{tet}† of a general cubic surface ϕ_3 through the vertices of the tetrahedron. The equation of the surface is

$$\begin{aligned}\phi_5 \equiv y_1^2\,[y_2^2 u + y_3^2 u' + y_4^2 u'' + A y_2 y_3 y_4] \\ + y_1[y_4^2 \phi_2 + y_2 y_3(B y_3 y_4 + y_2 u''')] \\ + y_2 y_3 y_4[C y_3 y_4 + D y_2 y_4 + E y_2 y_3] = 0,\end{aligned}$$

where u, u', u'', u''' are linear in (y_3, y_4), (y_2, y_4), (y_2, y_3), (y_3, y_4), respectively, and ϕ_2 is quadratic in (y_2, y_3), and where A, B, C, D and E are constants. The points whose coordinates are (1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1) are triple points, with non-composite tangent cones at each of the points.

3. *The Surface ϕ_5 with Three Ordinary Triple Points and a Tacnode.* The surface ϕ_5 is the transform by T_{tet} of a quartic surface with a double conic passing through three of the vertices of the tetrahedron and having the fourth vertex at a general point of the surface. The section of ϕ_5 by a plane through the tacnode and two triple points is a straight line and a pair of conics passing through these points.

The equation of ϕ_4 with a double conic is

$$\phi_4 \equiv [\, \textstyle\sum a_i x_j x_k]^2 - 4x_4^2\,[\psi_2 + x_4\psi_1] = 0,$$
$$(i, j, k = 1, 2, 3, 4, \cdots, i \neq j \neq k),$$

* Montesano, Napoli Rendiconti, (3), vol. 7 (1901), pp. 67–106.

† Hudson, *Cremona Transformations in Plane and Space*, Cambridge University Press, 1927, pp. 301–303.

where ψ_1 and ψ_2 are linear and quadratic, respectively, in the variables (x_1, x_2, x_3). The surface ψ_5 is

$$\phi_5 \equiv y_4^3\phi_1^2 - 4y_4\psi_4 - 4y_1y_2y_3\phi_2 = 0,$$

where ϕ_1 is linear in (y_1, y_2, y_3), ψ_4 is quadratic, and ϕ_2 linear in (y_2y_3, y_1y_3, y_1y_2). The point $(0, 0, 0, 1)$ is a tacnode on ϕ_5, and $(0, 0, 1, 0)$, $(0, 1, 0, 0)$, $(1, 0, 0, 0)$ are triple points with non-composite tangent cones.

4. *The Surface ϕ_5 with Three Ordinary Triple Points and a Tacnode, the Tacnode lying with one of the Triple Points on a Line of ϕ_5 Situated in the Tangent Plane at the Tacnode.* This surface is the transform by T_{tet} of a rational surface ϕ_4 of order four, having a double line and two double points in a plane through the double line. One fundamental point of the transformation is on the double line, two at the double points, and the fourth a general point of ϕ_4. The plane of the double points and double line is tangent to ϕ_4 along the line joining the double points. The transform of ϕ_4 by T_{tet} is

$$\phi_5 \equiv (y_2 - y_1)^2y_4^3 + (y_2 - y_1)^2y_4\psi_2 + y_4\psi_4 + \psi_5 = 0,$$

where ψ_2, ψ_4, and ψ_5 are forms in (y_1, y_2, y_3) of the order of their subscripts. The triple points are then $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, and the tacnode is $(0, 0, 0, 1)$.

5. *The Surface ϕ_5 with Two Triple Points and Two Tacnodes.* A space Cremona transformation of order three is defined by the web of surfaces of order three passing through three fixed conics k_2, k_2', k_2'' which lie in distinct planes, have one point in common, and meet by pairs in three points. The conjugate system is defined by cubic surfaces having in common three non-concurrent coplanar lines and a space cubic curve meeting each line once.

The inverse of this transformation carries a quadric surface through one of the lines and the intersection of the other two into a surface ψ_5 of order five having two of the fundamental conics as double curves and the third a simple curve. The transformation T_{tet} with fundamental points at the intersections of these conics carries ψ_5 into a surface ϕ_5 which has two triple points and two tacnodes. The tacnodes are the images of the double conics of ψ_5.

The equation ϕ_5 obtained by the above procedure is

$$\begin{aligned}\phi_5 \equiv\ & y_4^2\,[y_2y_3u_1 + Ay_1y_3u_2 + By_1y_2u_3 + Cy_1y_2y_3 \\ & + u_1(Dy_3u_2 + Dy_2u_3 + Ey_2y_3)] \\ & + y_4[y_2y_3u_4 + Ay_1y_3u_5 + By_1y_2u_6 + u_1(Fy_3u_5 + Dy_2u_6) \\ & + u_4(Fy_3u_2 + Dy_2u_3 + Ey_2y_3)] + u_4u_5u_6 = 0,\end{aligned}$$

where

$$u_1 \equiv b_4y_2 + \frac{b_3c_4}{c_2}y_3, \qquad u_4 \equiv y_2(b_1y_1 + b_3y_3) + \frac{b_3c_1}{c_2}y_1y_3,$$

$$u_2 \equiv a_4y_1 + \frac{a_3c_4}{c_1}y_3, \qquad u_5 \equiv y_1(a_2y_2 + a_3y_3) + \frac{a_3c_2}{c_1}y_2y_3,$$

$$u_3 \equiv a_4y_1 + \frac{a_2b_4}{b_1}y_2, \qquad u_6 \equiv y_1(a_2y_2 + a_3y_3) + \frac{a_2b_3}{b_1}y_2y_3;$$

and the double conics of the first transformation are

$$\begin{aligned} x_1 &= 0, \quad \sum a_ix_jx_k = 0, \quad (i, j, k = 2, 3, 4, \cdots, i \neq j \neq k), \\ x_2 &= 0, \quad \sum b_ix_jx_k = 0, \quad (i, j, k = 1, 3, 4, \cdots, i \neq j \neq k), \\ x_3 &= 0, \quad \sum c_ix_jx_k = 0, \quad (i, j, k = 1, 2, 4, \cdots, i \neq j \neq k). \end{aligned}$$

The points (0, 1, 0, 0) and (0, 0, 1, 0) are tacnodes and the points (1, 0, 0, 0), (0, 0, 0, 1) are triple points. The line $y_2=0$, $y_3=0$ lies on the surface.

6. *The Surface ϕ_5 with One Triple Point and Three Tacnodes.* The inverse of the first transformation of §5 carries a quadric surface through the intersections of the fundamental straight lines into a surface ψ_6 of order six having the fundamental conics as double curves, their common intersection a four-fold point, and the three points of intersection of these conics by pairs three-fold points. The transformation T_{tet}, with fundamental points at these multiple points, carries ψ_6 into a ϕ_5 with one triple point and three tacnodes. The procedure described above gives

$$\begin{aligned}\phi_5 \equiv\ & y_4^2\,[y_1y_2y_3 + u_1(y_3u_2 + y_2u_3 + y_2y_3) \\ & + y_1u_2(u_3 + y_3) + y_1y_2u_3] + y_4[u_1(y_3u_5 + y_2u_6) \\ & + u_4(y_3u_2 + y_2u_3 + y_2y_3) + y_1u_2u_6 + y_1u_5(u_3 + y_3) \\ & + y_1y_2u_6] + u_4(y_3u_5 + y_2u_6) + y_1u_5u_6 = 0,\end{aligned}$$

where u_i are as indicated in §5. The point (0, 0, 0, 1) is an ordinary triple point and the points (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0) are tacnodes. The tangent cones at the tacnodes are, respectively,

$$(a_2y_2 + a_3y_3 + a_4y_4)^2 = 0, \quad (b_1y_1 + b_3y_3 + b_4y_4)^2 = 0,$$
$$(c_1y_1 + c_2y_2 + c_4y_4)^2 = 0.$$

7. *The Surface ϕ_5 with an Ordinary Triple Point and a Tacnodal Triple Point.* A Cremona transformation is defined by the web of quadric surfaces containing a fixed conic and an arbitrary point. A special case of this transformation arises when the point is on the fixed conic.

If the conic of this web is tangent to a rational quartic surface with a double line at a general point of the line, the transformation defined by the web carries the quartic surface into a quintic having an ordinary triple point and a triple point with an adjacent infinitesimal double line,* or tacnodal triple point, both lying on the fundamental conic.

The equation of the quintic obtained by the above transformation is

$$\phi_5 \equiv y_4^2y_3^2\psi_1(y_1, y_2) + y_4y_3\phi_1(y_1, y_2, y_3)\cdot\phi_2(y_1, y_2) + y_3^2y_1\psi_2(y_1, y_2)$$
$$+ y_3\psi_4(y_1, y_2) + \psi_5(y_1, y_2) = 0.$$

The point (0, 0, 1, 0) is an ordinary triple point and (0, 0, 0, 1) a tacnodal triple point. In the plane $y_3=0$ are five straight lines on the surface, images of the residual intersections of the fundamental conic and the quartic surface. On ϕ_5 are eighteen conics passing through the triple points and lying by pairs in nine planes through the triple points.

8. *The Surface ϕ_5 with One Tacnodal Triple Point and One Tacnode.* In the following the vertices of the tetrahedron of reference are the points A_i, those of the conjugate system B_i, $(i=1, 2, 3, 4)$.

A rational surface of order four of the third type of Noether has a double point A_4 which is a cusp in a general plane section through it. The surface has a simple line passing through the double point; a general section through this line is a cubic curve

* Segre, Annali di Matematica, (2), vol. 25 (1896), pp. 1–53.

having the line as inflectional tangent at A_4. The tangent cone at A_4 is the plane p taken twice; this is tangent to ϕ_4 along the line. A section by this plane is a conic tangent to the line at A_4.

In the transformation of §7, let the conic be tangent to ϕ_4 at A_4. $T\equiv[F_2\equiv C_2A_4p,\ F_2'\equiv C_2'B_4p']$. Under $T, \phi_4\sim\phi_5::B_3^2B_4^3$, where B_3 is a tacnode, and B_4 is a tacnodal triple point. The conic C_2' is on ϕ_5. As in (7), the point B_4 is a tacnodal triple point, the tangent planes being p' taken twice, and p_1' containing C_2'.

The equation of the quintic surface obtained by this method is

$$\begin{aligned}\phi_5 \equiv\ & y_1y_3^2y_4^2 + 2y_3y_4[y_1y_3\phi_1 + \phi_3 - y_1(y_1y_2 + y_1^2 + y_2^2)] \\ & + (y_1y_2 + y_1^2 + y_2^2)[y_1(y_1y_2 + y_1^2 + y_2^2) - 2(y_1y_3\phi_1 + \phi_3)] \\ & - y_1^2y_3^3 + y_1y_3^2C_2 + y_1y_3C_3 + y_1C_4 = 0,\end{aligned}$$

where ϕ_1, ϕ_3, and C_i are forms in (y_1, y_2) of the order of their subscripts.

The point (0, 0, 0, 1) is a tacnodal triple point and (0, 0, 1, 0) is a tacnode lying on a line of ϕ_5 situated in the simple tangent plane at the triple point. The section of ϕ_5 by the simple tangent plane at the triple point is the conic $y_3y_4-y_2^2=0$ and the line $y_2=0$ counted three times; this line is the image of the simple line of ϕ_4.

9. *The Surface ϕ_5 with an Ordinary Triple Point and an Oscnode.* A quadratic Cremona transformation is defined by the quadric surfaces F_2 having in common two generators and osculating at their point of intersection.* Let the generators be l_1 and l_2 and their point of intersection A_1. The transformation is

$$T \equiv [F_2::l_1, l_2, A_1, F_2' \equiv l_1', l_2', B_1].$$

Let A_1 be a generic point on a monoidal quartic surface ϕ_4, and let l_1, l_2 each osculate ϕ_4 at A_1. Let the triple point of ϕ_4 be A_4. Under $T, \phi_4\sim\phi_5$ with an ordinary triple point at B_4 and an oscnode at B_1.

A general straight line through A_1 has three residual intersections with ϕ_4; hence the image straight line has three intersections with ϕ_5 not at B_1. A general straight line meets ϕ_4 in

* Hudson, loc. cit., pp. 197–198.

four points; hence its image conic has four points in common with ϕ_5 not at B_1; that is, at B_1 are three consecutive double points on the image conic or B_1 is an oscnode on ϕ_5. The equation of ϕ_4 is

$$\phi_4 \equiv \psi_3 x_4 + \psi_4 = 0,$$

where

$$\psi_3 \equiv A x_1^3 + w_1 x_1^2 + w_2 x_1 + w_3,$$
$$\psi_4 \equiv B x_1^2 x_2 x_3 + u_3 x_1 + u_4,$$

and where w_i, u_i are binary forms in (x_2, x_3) of order i. The line $x_2=0$, $x_4=0$ osculates ϕ_4 at $(1, 0, 0, 0)$. The equations of transformation are

$$\rho x_1 = B(y_1y_4 - y_2y_3), \quad \rho x_2 = A y_2 y_4, \quad \rho x_3 = y_3 y_4, \quad \rho x_4 = y_4^2.$$

The equation of the resulting quintic is

$$B^2(y_1y_4 - y_2y_3)^2[ABy_1 + w_1] + B(y_1y_4 - y_2y_3)[w_2y_4 + u_3] + y_4[w_3y_4 + u_4] = 0,$$

where w_i, u_i, are the above forms in (y_2, y_3). The point $(0, 0, 0, 1)$ is a triple point and $(1, 0, 0, 0)$ is an oscnode.

10. *The Surface ϕ_5 with an Oscnode and a Tacnode.* Applying the transformation of §9 to a quartic surface of the first type of Noether, we find

$$\phi_5 \equiv B^2(y_1y_4 - y_2y_3)^2[ABy_1 + y_4 + u_1] + B(y_1y_4 - y_2y_3)[u_2y_4 + u_3] + u_4y_4 = 0,$$

where u_i are binary forms in the variables (y_2, y_3) of order i. The point $(0, 0, 0, 1)$ is a tacnode and $(1, 0, 0, 0)$ an oscnode.

11. *The Surface ϕ_5 with an Oscnode and a Double Point of the First Order.* Such a surface is the transform by the above transformation of a quartic of the second type of Noether. The equation of this quintic is

$$\begin{aligned}\phi_5 \equiv\ & \phi_1^2 y_4^3 + 2y_4^2[y_3\phi_1(y_3 + \psi_1) + \phi_3] \\ & + y_4\{y_3^4 + 2y_3^3\psi_1 - [D\psi_1 + F\phi_1 + D(2Fy_1 + Gy_2) \\ & + BK_1y_1 + K_2y_2]y_2y_3^2 + y_2y_3\phi_2 + y_2^3\phi_1'\} \\ & + y_2y_3\psi_3 + AB^3y_1y_2^2y_3^2 = 0,\end{aligned}$$

where

$$D=b_1B, \quad E=b_2A, \quad F=c_1B, \quad G=c_2A, \quad \phi_1=Dy_1+Ey_2,$$

and where

$$\psi_1 = Fy_1 + (G - D)y_2,$$
$$\phi_3 = AB^3y_1^3 + B^2K_3y_1^2y_2 + BK_4y_1y_2^2 + K_5y_2^3,$$
$$\phi_2 = -2AB^3y_1^2 + B(K_6 - 2BK)y_1y_2 - BK_4y_2^2,$$
$$\phi_1' = BK_7y_1 + (K_8 - BK_4)y_3,$$
$$\psi_3 = (2DF - BK_1)y_2y_3^2 + B(BK_3 - K_6)y_2^2y_3 - 2Fy_3^3 - BK_7y_2^3.$$

The point (1, 0, 0, 0) is an oscnode, and (0, 0, 0, 1) is a double point of the first order.

12. *The Surface ϕ_5 with an Oscnode and a Double Point of the Second Order.* Such a surface is the transform of a quartic surface of the third type of Noether. The equation of this quintic is

$$\phi_5 \equiv B^2(y_1y_4 - y_2y_3)^2\phi_1 + B(y_1y_4 - y_2y_3)[\phi_3 + y_2y_4\psi_1] + y_2^2y_4[Ay_2y_4 + \phi_2] = 0,$$

where ϕ_1 is linear in all variables, and where ψ_1, ϕ_2, ϕ_3 are forms in (y_2, y_3) of the order of their subscripts. The point (1, 0, 0, 0) is an oscnode, and (0, 0, 0, 1) is a double point of the second order.

13. *The Surface ϕ_5 as a General Member of a Homaloidal Family of a Cremona Space Transformation.* The quintic surfaces discussed above are all rational and their (1, 1) representations on a plane π are known. Cremona* has shown that the set of space transformations having ϕ as a general member of the first homaloidal family corresponds to a set of plane Cremona transformations in π.

If, by a Cremona transformation, ϕ_n is the transform of a rational ψ_m which is a general member of the homaloidal families of k Cremona transformations, then ϕ_n will serve as the general member of the homaloidal families of k Cremona transformations.

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* Cremona, Istituto Lombardo Rendiconti, (2), vol. 4 (1871), pp. 269–279; and Annali di Matematica, (2), vol. 5 (1871), pp. 131–162.

14. *Conclusion*. Each of the rational quintic surfaces discussed above can serve as a general member of the homaloidal family of a Cremona transformation. The surfaces of §§5, 6, 9 are the transforms, by Cremona transformations, of a general quadric surface. The surface of §2 is the transform of a general ϕ_3; that of §10 is the transform of a quartic surface of the first type of Noether which is the transform of a general ϕ_3. The surfaces of §§8, 12 are the transforms of a quartic surface of the third type of Noether which is the transform of a special quartic of the first type of Noether with a double point in the tangent plane at the tacnode; this plane is tangent to the surface along the line joining the double points,* and this surface is a transform of a general ϕ_3. The surfaces of §§7, 11 are transforms of a quartic with a double line.† The surface of §4 is the transform of a quartic with a double line and two double points coplanar with the double line; the homaloidal family of which this ϕ_4 is a general member have in common the double line and also have contact along the line joining the double points. The surface of §3 is the transform of a quartic with a double conic.‡

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* Noether, Mathematische Annalen, (3), vol. 33 (1889), pp. 546–571.

† Montesano, Roma Rendiconti, (4), vol. 5–2 (1889), pp. 123–130.

‡ Aroldi, Giornale di Matematiche, (3), vol. 11 (1920), pp. 175–192.