All intersector sequences of order three will be closed if the invariant

$$p^2(p_{21}^{2}\alpha^2 + p_{12}^{2}\beta^2) + 2p_{12}^{3}p_{21}^{3}\alpha^2\beta^2$$

vanishes. This can happen either if $p=0$, $\alpha=0$ (or $\beta=0$), or $\alpha=0$, $\beta=0$. In the first case one branch of the flecnode curve of R_{yz} is plane, $(\alpha=0)$, and $R_{\psi\phi}$ degenerates into the tangents of a plane curve. In the second case both branches of the flecnode curve of R_{yz} are plane and $R_{\psi\phi}$ degenerates into a straight line.

It is obvious that in the preceding developments the order of the lines l_{yz}, $l_{\psi\phi}$, $l_{\eta\theta}$ can be reversed without in any way affecting results. The analysis would be based upon a system of first-order equations of the same type as (4), (5) and obtainable from (4), (5) by simple processes.

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NOTE ON THE REDUCIBILITY OF ALGEBRAS WITHOUT A FINITE BASE*

BY M. H. INGRAHAM

It is the purpose of this note to discuss the reducibility of linear associative algebras which are not assumed to possess a finite base. J. H. M. Wedderburn,† in seeking to generalize certain theorems on the structure of an algebra, has considered algebras in which restrictions are placed upon the character of the idempotent elements. The summations involved in his study need not be finite. This seems to be one natural line of attack.

J. W. Young‡ has approached the subject from the point of view of the groups involved. His definition of a finite algebra is, however, unsatisfactory, not being sufficiently restrictive.

I have studied infinite algebras in connection with the results that can be obtained by a use of the "axiom of choice" and the theory of transfinite ordinals. This note, however, does not

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† J. H. M. Wedderburn, *Algebras which do not possess a finite base*, Transactions of this Society, vol. 26 (1924), pp. 395–426.

‡ J. W. Young, *A new formulation for general algebras*, Annals of Mathematics, vol. 29 (1927), pp. 47–60. See particularly p. 60.

assume the existence of an infinite base for an algebra but merely the usual postulates for a finite linear associative algebra $A = [a]$ over a field $\Xi = [\xi]$, where the postulate requiring the existence of a finite base is replaced by the following extremely mild postulate: for every pair of elements a_1 and a_2 in the algebra A, we have $1 \cdot a_1 = a_1$ and $a_1 + a_2 \cdot 0 = a_1$, where the elements 1 and 0 are the unit and zero elements respectively of the field Ξ over which the algebra A is taken. There is no difficulty in generalizing the elementary properties of 0 and of a principal unit if it exists.

As in the finite case, a set $S = [s]$ of elements of A is said to be a linear set if for every pair of scalars ξ_1, ξ_2 in Ξ and elements s_1 and s_2 in S, $\xi_1 s_1 + \xi_2 s_2$ is in S. If C is a class of linear sets of elements of A, then the sum of these linear sets is defined as the least linear set containing all the linear sets of C. This is equivalent to defining the sum as the totality of finite linear combinations with scalar coefficients of the elements of the sets of C. The product $S_1 S_2$ of two linear sets S_1 and S_2 is the least linear set containing every element of the form $s_1 s_2$, where s_1 and s_2 are in S_1 and S_2 respectively. I have discussed elsewhere* other properties of linear sets.

As in the case of finite algebras, an invariant proper sub-algebra A_1 of A is one for which both $A_1 A$ and $A A_1$ are contained in A_1. The sum and intersection of a class of invariant sub-algebras is an invariant sub-algebra (or zero) and the sum of two distinct maximal invariant sub-algebras is A. Moreover, we may define the sum of two or more algebras $A_1, A_2, \cdots$ as a direct sum which we denote $A_1(+)A_2(+)\cdots$, or $(\Sigma)A_i$, if for every $i \neq j$, $A_i A_j = A_j A_i = 0$, and the intersection $A_i \wedge A_j = 0$. If A has a principal unit e and $A = A_1(+)A_2$, then A_1 has a principal unit e_1, and A_2 has a principal unit e_2, where $e = e_1 + e_2$. We may prove, as in the finite case, the following theorem.

THEOREM 1. *If A_1 and A_2 are sub-algebras of A either of which has a principal unit, and if $A_1 A_2 = A_2 A_1 = 0$, then $A_1 \wedge A_2 = 0$ and $A_1(+)A_2$ is a direct sum.*

THEOREM 2. *If A has an invariant sub-algebra A_1 which possesses a principal unit e_1, then A is reducible and has A_1 for one component.*

* M. H. Ingraham, *A general theory of linear sets*, Transactions of this Society, vol. 27 (1925), pp. 163–196.

J. H. M. Wedderburn, in his above mentioned paper, proves this theorem on the basis of a postulate which requires that for every linear set there exists a supplementary set such that the sum of the two is A and the intersection zero. His proof, however, makes no essential use of this hypothesis. The first real difference in this theory from that of finite algebras comes in attempting to generalize the following theorem.*

Any reducible finite algebra A with a principal unit e can be expressed as the direct sum of irreducible algebras each with a principal unit, in one way and only one way apart from the arrangement of the component algebras.

This theorem is not true in the extended theory for there exist algebras of infinite order with principal units which are not expressible as the direct sum of irreducible algebras. Although the existence of a complete reduction of a reducible algebra can not be proved in general, it will be seen as an immediate corollary of Theorem 4 that the uniqueness, except for order of components of any complete reduction into sub-algebras each with a principal unit, does hold in general. This situation is given by Theorems 3 and 4.

THEOREM 3. *If A can be expressed as the direct, sum of sub-algebras in two ways $A=(\Sigma)A_i=(\Sigma)B_i$, such that $AA_i=A_iA=A_{ij}$ and $B_iA=AB_i=B_i$, then each A_i and B_i can be expressed as the direct sum of one or more sub-algebras $A_i=(\Sigma)A_{ij}$, $B_i=(\Sigma)B_{ij}$ such that apart from order the A_{ij}'s and the B_{ij}'s are identical.*

THEOREM 4. *If A has a principal unit, it can not be expressed as the direct sum of more than a finite number of sub-algebras.*

PROOF OF THEOREM 3. Let $A_{ij}=B_{ji}$ equal the intersections of A_i and B_j.

$A_i=A_iA=\Sigma_jA_iB_j$. But A_iB_j is a portion of the intersection of A_i and B_j, and hence $A_i=\Sigma_jA_{ij}$ and, moreover, it is a direct sum since A_{ij} is in B_j and for every $j\neq k$, $B_jB_k=B_kB_j=0$ and $(B_j\wedge B_k)=0$. Similarly, $B_j=(\Sigma)_iB_{ji}$. Hence $A=(\Sigma)_iA_i=(\Sigma)_{ij}A_i$ $=(\Sigma)_j(B_j)=(\Sigma_{ij}B_{ji})$, and apart from order, the A_{ij} and the B_{ji} are identical. Moreover, it should be noted that $A_{ij}=A_iB_j$ $=B_jA_i$.

* Dickson, *Algebras and their Arithmetics*, p. 35.

COROLLARY. *If an algebra A can be reduced to the direct sum of irreducible algebras A_i, each A_i having the property that $A_iA = AA_i = A_i$, this reduction is, apart from order, unique.*

It is obvious that $AA_i = A_iA = A_i$ if A_i contains a principal unit. Hence reduction of an algebra to irreducible subalgebras each with a principal unit is unique.

In order to prove Theorem 4, consider $A = (\Sigma)A_i$ and containing a principal unit e. The principal unit e can be expressed uniquely as a sum of a finite number of elements not more than one of which belongs to any A_i. Then $e = \sum a_i$, where, except for a finite number of values of i, $a_i = 0$ if A_j is such that $a_j = 0$; then $A_j = eA_j = 0$, and the theorem follows at once.

The following three examples somewhat clarify the situation. Example 1 is an algebra which, although it contains no principal unit, is the direct sum of an infinite number of irreducible subalgebras each with a principal unit and hence, by the corollary to Theorem 4, is uniquely reducible. Example 2 is an algebra with a principal unit containing Example 1 and its invariant sub-algebras as invariant sub-algebras, and hence by Theorem 4 is not reducible to the direct sum of irreducible sub-algebras. Example 3 is the most interesting of the group, being an algebra such that every invariant sub-algebra contains a proper invariant sub-algebra with a principal unit, and hence is reducible. All of these algebras contain denumerable finitely linearly independent bases and hence we do not believe that any simple postulates as to bases for the infinite algebras will yield appreciably greater results in this portion of the theory. Although all the examples are commutative, non-commutative examples can easily be built up as the direct product of these and finite non-commutative division algebras.

EXAMPLE 1. Let $A =$ [all sequences $\{\xi_i\}$ of scalars which are zero except for a finite number of elements]. Let addition, multiplication be the addition and multiplication of corresponding elements, and scalar multiplication the multiplication of each element by the given scalar. Let A_i be all the sequences for which $\xi_j = 0$, $(i \neq j)$. Then $A = (\Sigma)A_i$.

EXAMPLE 2. Let $A =$ [all sequences $\{\xi_i\}$ of scalars such that $\xi_i = \xi_{i+1}$ except for a finite set of values for i]. Let addition, multiplication, and scalar multiplication be defined as above.

Each A_i of Example 1 is an invariant sub-algebra of A but $A \neq (\Sigma) A_i$.

EXAMPLE 3. Consider $u_0 = u_1 + v_1$, where

$$u_0^2 = u_0, \qquad u_1^2 = u_1, \qquad v_1^2 = v_1,$$

$$u_0 u_1 = u_1 u_0 = u_1, \qquad u_1 v_1 = 0, \qquad u_0 v_1 = v_1 u_0 = v_1.$$

In a recursive manner, let

$$u_1 = u_2 + v_2, \qquad v_1 = u_3 + v_3,$$

where the table of multiplication for u_1, u_2, v_2 can be gotten from that for u_0, u_1 and v_1 by replacing u_0 by u_1, u_1 by u_2, v_1 by v_2, and the multiplication table for v_1, u_3 and v_3 can be gotten in a similar manner by replacing u_0 by v_1, u_1 by u_3, and v_1 by v_3. In a similar manner, express each of u_2, v_2, u_3 and v_3 as sums of a pair of elements, etc. If we call u_1 and v_1 direct descendents of u_0, u_2 and v_2 direct descendents of u_1 and descendents of u_0, etc., the table of multiplication can be stated in the following geneological fashion. Each u or v is idempotent. An element times its descendent is the descendent; an element times any brother or cousin no matter how far removed is zero. It is also clear that the u's form a base for the total algebra. Moreover, if any v or u is in an invariant sub-algebra A_1 of A, it and all its descendents form a base for an invariant sub-algebra A_2 of A_1 and A of which it is the principal unit. Suppose then A_1 contains an element $a_1 = \sum_{i=1}^{n} \xi_i u_i$. Let u_n be a u in the lowest generation represented in a_1. Then $a_1 u_n = (\sum_1^n \xi_i) u_n$. If $\sum_1^n \xi_i \neq 0$, A_1 contains u_n. If $\sum_1^n \xi_i = 0$ and if w is the direct ancestor of u_n, then $a_1 w = \sum_1^{n-1} \xi_i w + \xi_n u_n = -\xi_n v_n$ and A_1 contains v_n, hence A_1 contains either u_n or v_n, and hence each of their descendents and hence a proper invariant sub-algebra with a principal unit.

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