A TRIAD OF RULED SURFACES DEFINED BY RECIPROCAL POLARS*

BY A. F. CARPENTER

As an application of the theory of triads of ruled surfaces[†] in projective differential geometry, we shall consider here a system defined as follows. Let l_{yz} be any line element of a general ruled surface R_{yz}, the points P_y, P_z being the flecnodes of l_{yz}. The defining system of differential equations for R_{yz} will have the form

$$
\begin{aligned}
y'' + p_{12}z' + q_{11}y + q_{12}z &= 0, \\
z'' + p_{21}y' + q_{21}y + q_{22}z &= 0,
\end{aligned}
\tag{1}
$$

where differentiation is with respect to a parameter x and where $p_{12}' = 2q_{12}$, $p_{21}' = 2q_{21}$.

The planes osculating the flecnode curve C of R_{yz} at P_y, P_z intersect in a line $l_{\psi\phi}$, the points P_ψ, P_ϕ of which are those in which $l_{\psi\phi}$ is cut by the tangents to C at the respective points P_z, P_y. The expressions for ψ, ϕ are

$$
\psi = p_{12}z' + q_{12}z, \quad \phi = p_{21}y' + q_{21}y.
\tag{2}
$$

The polar reciprocal of $l_{\psi\phi}$ with respect to the linear complex which osculates R_{yz} along l_{yz} is the line $l_{\eta\theta}$, the points P_η, P_θ of which are given by the expressions

$$
\eta = p_{12}z' + p_{12}p_{21}y + q_{12}z, \quad \theta = p_{21}y' + q_{21}y + p_{12}p_{21}z.
\tag{3}
$$

The points P_y, P_ψ, P_η are collinear, as are also the points P_z, P_ϕ, P_θ.

A set of three lines l_{yz}, $l_{\psi\phi}$, $l_{\eta\theta}$, as thus defined, corresponds to each value of the parameter x and determines in this way three ruled surfaces R_{yz}, $R_{\psi\phi}$, $R_{\eta\theta}$. From (1), (2) and (3) we obtain the defining system of differential equations for this triad of ruled surfaces.[‡] It is

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$$y' = -r_{21}y + \pi_{21}\phi, \quad \psi' = 3r_{12}\psi + (\alpha - p_{12})\phi - \alpha\theta,$$

$$z' = -r_{12}z + \pi_{12}\psi, \quad \phi' = (\beta - p_{21})\psi + 3r_{21}\phi - \beta\eta,$$

$$(4) \quad y = \pi_{12}\pi_{21}(\eta - \psi), \quad \eta' = py - p_{12}p_{21}\alpha z + 3r_{12}\eta,$$

$$z = \pi_{12}\pi_{21}(\theta - \phi), \quad \theta' = -p_{12}p_{21}\beta y - pz + 3r_{21}\theta,$$

where*

$$p_{12}^2 p_{21}\alpha = p_{12}^2 q_{22} - p_{12}q_{12}' + 3q_{12}^2,$$

$$(5) \quad p_{12}p_{21}^2\beta = p_{21}^2 q_{11} - p_{21}q_{21}' + 3q_{21}^2,$$

$$p = p_{12}q_{21} - p_{21}q_{12}, \quad \pi_{12}p_{12} = \pi_{21}p_{21} = 1,$$

$$p_{12}r_{12} = q_{12}, \quad p_{21}r_{21} = q_{21}.$$

The three lines l_{yz}, $l_{\psi\phi}$, $l_{\eta\theta}$ determine a quadric Q. On each of these lines there will be two points such that tangents at these points to the respective surfaces R_{yz}, $R_{\psi\phi}$, $R_{\eta\theta}$ will be lines of that regulus of Q whose lines intersect l_{yz}, $l_{\psi\phi}$, $l_{\eta\theta}$. These points are found by factoring the respective quadratic covariants†

$$(6) \quad p_{21}y^2 - p_{12}z^2, \quad \beta\psi^2 - \alpha\phi^2, \quad \beta\eta^2 + 2p\pi_{12}\pi_{21}\eta\theta - \alpha\theta^2.$$

In this connection we remark that *the first of these covariants determines the complex points of l_{yz}.*‡

On each of the lines l_{yz}, $l_{\psi\phi}$, $l_{\eta\theta}$ lie two points at which intersector tangents§ to the surfaces R_{yz}, $R_{\psi\phi}$, $R_{\eta\theta}$ will also be tangents to the respective surfaces $R_{\psi\phi}$, $R_{\eta\theta}$, R_{yz}. These points are given by the respective covariants¶

$$(7) \quad \begin{array}{c} (p_{21}\alpha - p_{12}\beta)yz, \quad p(\beta\psi^2 - \alpha\phi^2), \\ p\pi_{12}\eta^2 + (p_{21}\alpha - p_{12}\beta)\eta\theta - p\pi_{21}\theta^2. \end{array}$$

If the invariant $p_{21}\alpha - p_{12}\beta \neq 0$, then from the first expression in (7) it follows that the intersector tangents to R_{yz} at the flecnodes of

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* We shall assume that p_{12} and p_{21} do not vanish identically, that is, that neither branch of the flecnode curve of R_{yz} is a straight line.

† T, p. 269 et seq.

‡ A. F. Carpenter, *Ruled surfaces whose flecnode curves have plane branches*, Transactions of this Society, vol. 16 (1915).

§ E. P. Lane, *Ruled surfaces with generators in one-to-one correspondence*, Transactions of this Society, vol. 25 (1923).

¶ T, p. 269 et seq.

l_{yz} are tangent to $R_{\psi\phi}$. If $p_{21}\alpha - p_{12}\beta \equiv 0$, then the intersector tangents to R_{yz} at all points of l_{yz} are tangent to $R_{\psi\phi}$. This conclusion can be verified in the following way.

Let $u = y + \lambda z$ be the expression for the general point on l_{yz} and let us choose for system of reference the tetrahedron $P_y P_z P_\rho P_\sigma$, where the unit point is so chosen that all coordinates of the four vertices are zero except $y_1 = z_2 = \rho_3 = \sigma_4 = 1$, and where

$$(8) \qquad \rho = 2y' + p_{12}z, \quad \sigma = 2z' + p_{21}y.$$

Then with the help of equations (4) and (8) the equation of the tangent plane to R_{yz} at P_u is found to be

$$(9) \qquad \lambda x_3 - x_4 = 0.$$

This plane cuts $l_{\psi\phi}$ in the point P_v, where

$$v = \psi + k\phi, \quad p_{21}k\lambda = p_{12}.$$

The equation of the tangent plane to $R_{\psi\phi}$ at P_v is, similarly,

$$p_{21}\alpha\lambda x_1 - p_{12}\beta x_2 - (p_{12}^2\beta + 2q_{21}\alpha\lambda)x_3 + (p_{21}^2\alpha\lambda + 2q_{12}\beta)x_4 = 0,$$

and the point P_u lies on this plane for all values of λ, if, and only if, $p_{21}\alpha - p_{12}\beta \equiv 0$.

In a previous paper by the author* it was proved that the osculating planes of the flecnode curve, the complex curve and the harmonic curve of R_{yz} at the points in which these curves cut any line element l_{yz}, will form an axial pencil if, and only if, $p_{21}\alpha - p_{12}\beta = 0$. In view of what precedes we may state the following theorem.

THEOREM 1. *If the planes osculating the flecnode curve, the complex curve, and the harmonic curve of a general ruled surface R at the points of intersection of these curves with each line element of R form an axial pencil, then R and the ruled surface S generated by the axis of this pencil of planes are the focal surfaces of the congruence of intersector tangents of R (or S) with respect to S(or R).*

If the invariant $p \not\equiv 0$, then from (6) and (7) it results that the two points of $l_{\psi\phi}$ at which lines of Q are tangent to $R_{\psi\phi}$ are also the points at which intersector tangents of $R_{\psi\phi}$ are tangent to $R_{\eta\theta}$. If $p \equiv 0$, then the intersector tangents to $R_{\psi\phi}$ at all points of $l_{\psi\phi}$ are

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* A. F. Carpenter, *A theorem on ruled surfaces*, this Bulletin, vol. 34 (1928), pp. 479–481.

tangent to $R_{\eta\theta}$. This can be verified as above. But if $p \equiv 0$, then R_{yz} is identically self-dual.* We can sum up these results in the following statement.

THEOREM 2. *Let the planes osculating the flecnode curve of a general ruled surface R at the two points of intersection of this curve with a line element l of R intersect in s, and let t be the polar reciprocal of s with respect to the linear complex osculating R along l. Then if R is identically self-dual, the two ruled surfaces S and T generated by s and t will be the focal surfaces of the congruence of intersector tangents of S (or T) with respect to T (or S).*

If both $p_{21}\alpha - p_{12}\beta \equiv 0$ and $p \equiv 0$, then the third covariant of (7) vanishes identically. Under these conditions, R_{yz} is a quadric.†

Two of the intersector tangents of R_{yz} at points of l_{yz} must be asymptotic tangents of R_{yz}. We find these points as follows. In order that the curve C_u generated by P_u be an asymptotic curve of R_{yz}, λ must satisfy the differential equation

$$(10) \qquad\qquad 2\lambda' + p_{21}\lambda^2 - p_{12} = 0.$$

By the use of (4) and (10), we find

$$(11) \quad u' = - r_{21}y - \tfrac{1}{2}(p_{21}\lambda^2 + 2r_{12}\lambda - p_{12})z + \pi_{12}\lambda\psi + \pi_{21}\phi.$$

Now the asymptotic tangent at P_u will cut $l_{\psi\phi}$ if $u' + \mu u$ is linear in ψ and ϕ. It follows that

$$u - r_{21} = 0, \quad p_{21}\lambda^2 - 2\mu\lambda + 2r_{12}\lambda - p_{12} = 0,$$

and therefore

$$(12) \qquad\qquad p_{21}\lambda^2 - 2(r_{21} - r_{12})\lambda - p_{12} = 0.$$

The values of λ from (12) determine the two values of u sought. The corresponding quadratic covariant is found to be

$$(13) \qquad p_{21}y^2 + 2(r_{21} - r_{12})yz - p_{12}z^2 = \pi_{21}PQ,$$

where

$$P = p_{21}y + Uz, \; Q = p_{21}y + Vz, \; U = r_{21} - r_{12} + r, \; V = r_{21} - r_{12} - r,$$
$$r = [(r_{21} - r_{12})^2 + p_{12}p_{21}]^{1/2}.$$

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* E. J. Wilczynski, *Projective Differential Geometry*, p. 144; p is a factor of the invariant.

† Wilczynski, loc. cit., p. 150.

The two points in which these asymptotic tangents cut $l_{\psi\phi}$ are given by $\pi_{12}\lambda_1\psi + \pi_{21}\phi$, $\pi_{12}\lambda_2\psi + \pi_{21}\phi$, where λ_1, λ_2 are the roots of (12), and the corresponding covariant is found to be

$$(14) \qquad p_{21}\psi^2 - 2(r_{21} - r_{12})\psi\phi - p_{12}\phi^2 = \pi_{12}CD,$$

where $C = p_{21}\psi - V\phi$, $D = p_{21}\psi - U\phi$.

By proceeding as above we find, for the points of l_{yz} at which asymptotic tangents to R_{yz} cut $l_{\eta\theta}$, the covariant

$$(15) \qquad p_{21}y^2 - 2(r_{21} - r_{12})yz - p_{12}z^2 = \pi_{21}RS,$$

where $R = p_{21}y - Uz$, $S = p_{21}y - Vz$, and for the points in which these tangents cut $l_{\eta\theta}$ the covariant

$$(16) \qquad p_{21}\eta^2 + 2(r_{21} - r_{12})\eta\theta - p_{12}\theta^2 = \pi_{21}AB,$$

where $A = p_{21}\eta + V\theta$, $B = p_{21}\eta + U\theta$.

By comparison of (13), (14), (15), (16) it results that *the line l_{yz} is cut by those faces of the tetrahedron $ABCD$ opposite the vertices A, B, C, D in the respective points P, Q, R, S. It is also seen that P_y, P_z separate harmonically both pairs of points P, R and Q, S.*

Since $r_{21} - r_{12} \equiv 0$ implies $p \equiv 0$, and conversely, it results that if R_{yz} is identically self-dual the asymptotic tangents to R_{yz} at the complex points of l_{yz} intersect both $R_{\psi\phi}$ and $R_{\eta\theta}$ and are thus rulings of the quadric determined by l_{yz}, $l_{\psi\phi}$, $l_{\eta\theta}$.

There exist, in general, for the three lines l_{yz}, $l_{\psi\phi}$, $l_{\eta\theta}$, two closed intersector sequences of order one.* The pairs of points in which these lines are cut by these sequences are given by the respective covariants

$$\begin{aligned}
& p_{21}\alpha\beta y^2 + p(\pi_{12}\alpha + \pi_{21}\beta)yz - p_{12}\alpha\beta z^2, \\
(17) \quad & p_{21}\alpha\beta\psi^2 - p(\pi_{12}\alpha + \pi_{21}\beta)\psi\phi - p_{12}\alpha\beta\phi^2, \\
& p_{12}\beta^2\eta^2 + p(\pi_{12}\alpha + \pi_{21}\beta)\eta\theta - p_{21}\alpha^2\theta^2.
\end{aligned}$$

The existence of a closed intersector sequence of order two is assured by the vanishing of the invariant

$$(18) \qquad p\pi_{12}\pi_{21}(p_{21}\alpha - p_{12}\beta),$$

and this closure property holds for all such sequences. From (18) it follows that, *under the conditions of either Theorem 1 or Theorem 2, all intersector sequences of order two are closed.*

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* T, pp. 272, 273.

All intersector sequences of order three will be closed if the invariant

$$p^2(p_{21}^2\alpha^2 + p_{12}^2\beta^2) + 2p_{12}^3p_{21}^3\alpha^2\beta^2$$

vanishes. This can happen either if $p=0$, $\alpha=0$ (or $\beta=0$), or $\alpha=0$, $\beta=0$. In the first case one branch of the flecnode curve of R_{yz} is plane, $(\alpha=0)$, and $R_{\psi\phi}$ degenerates into the tangents of a plane curve. In the second case both branches of the flecnode curve of R_{yz} are plane and $R_{\psi\phi}$ degenerates into a straight line.

It is obvious that in the preceding developments the order of the lines l_{yz}, $l_{\psi\phi}$, $l_{\eta\theta}$ can be reversed without in any way affecting results. The analysis would be based upon a system of first-order equations of the same type as (4), (5) and obtainable from (4), (5) by simple processes.

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NOTE ON THE REDUCIBILITY OF ALGEBRAS WITHOUT A FINITE BASE*

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It is the purpose of this note to discuss the reducibility of linear associative algebras which are not assumed to possess a finite base. J. H. M. Wedderburn,† in seeking to generalize certain theorems on the structure of an algebra, has considered algebras in which restrictions are placed upon the character of the idempotent elements. The summations involved in his study need not be finite. This seems to be one natural line of attack.

J. W. Young‡ has approached the subject from the point of view of the groups involved. His definition of a finite algebra is, however, unsatisfactory, not being sufficiently restrictive.

I have studied infinite algebras in connection with the results that can be obtained by a use of the "axiom of choice" and the theory of transfinite ordinals. This note, however, does not

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